7.5 Hyperbolas
7.5.1 Exercises
page 541(553): 3, 10, 14
10 Foundations of Trigonometry
10.1 Angles and their Measure
10.1.2 Exercises
page 709 (721): 9, 15, 17, 30, 35, 39, 41, 50

10.2 The Unit Circle: Cosine and Sine
10.2.2 Exercises
page 736 (748): 2, 7, 15, 21, 28, 31, 49, 50, 55
10.3 The Six Circular Functions and Fundamental Identities
10.3.2 Exercises
page 759 (771): 1, 7, 11, 21, 35, 59, 79, 86, 91, 129
10.4 Trigonometric Identities
10.4.1 Exercises
page 782 (794): 3, 14, 22a, 32, 43, 49

---

14 required textbook sections
7 class meetings
2 or 3 sections per class meeting

A diller, a dollar,
a witless trig scholar
on a ladder against a wall.
Should length over height
make an angle too slight,
the cosecant may prove his
downfall.

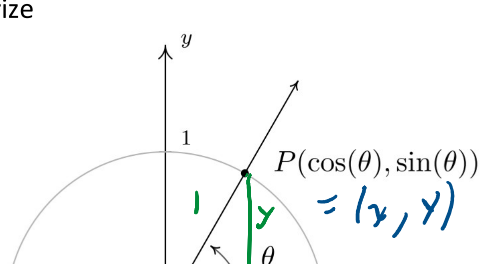
Copilot

Background and Analysis

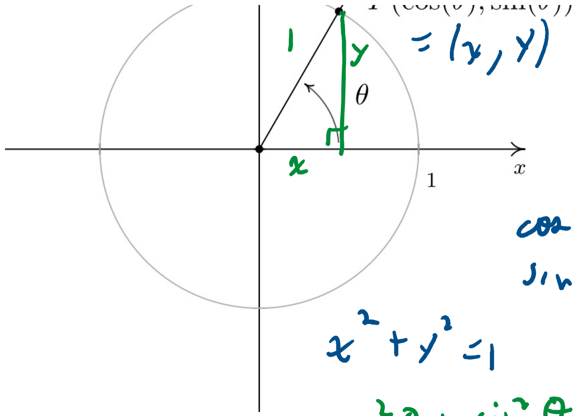
- **Original rhyme:** The first line, *"A dillar, a dollar,"* comes from a well-known nursery rhyme dating back to the 19th century:
  - A diller, a dollar,
    A ten o'clock scholar,
    What makes you come so soon?
    You used to come at ten o'clock,
    But now you come at noon.

10.2

Memorize



unit circle
center (0, 0)
radius 1

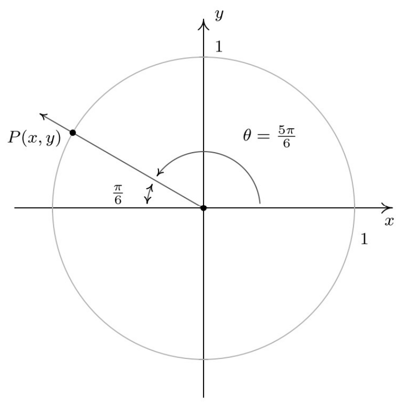


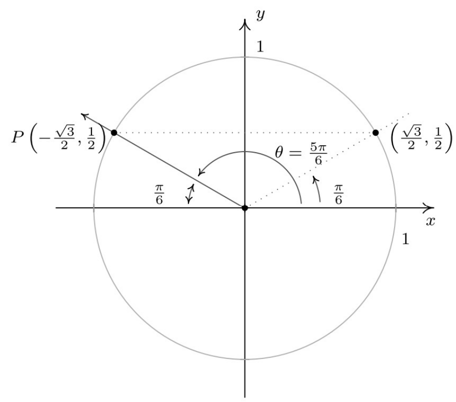
center (0, 0)
radius 1

$\cos\theta = x$
$\sin\theta = y$

$x^2 + y^2 = 1$

$\cos^2\theta + \sin^2\theta = 1$ } memorize
Pythagorean identity





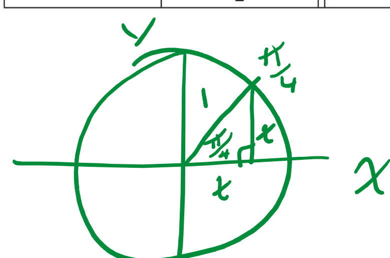
Memorize

Theorem 10.2. Reference Angle Theorem. Suppose α is the reference angle for θ. Then $\cos(\theta) = \pm\cos(\alpha)$ and $\sin(\theta) = \pm\sin(\alpha)$, where the choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

Memorize

Cosine and Sine Values of Common Angles

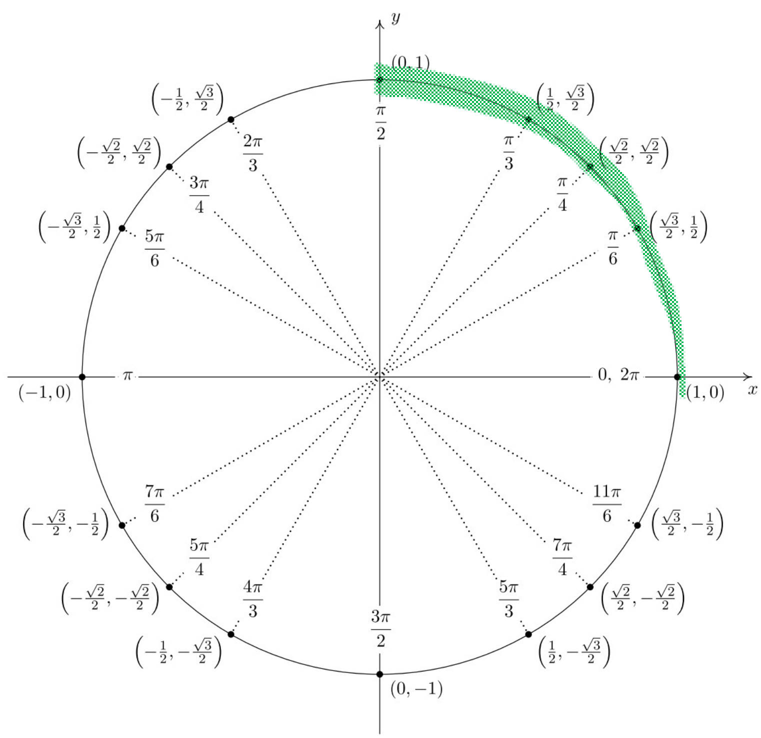
θ(degrees)	θ(radians)	$\cos(\theta)$	$\sin(\theta)$
0°	0	1	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90°	$\frac{\pi}{2}$	0	1



$x^2 + x^2 = 1^2$

$$x^2+x^2=1^2$$
$$2x^2=1$$
$$x^2=\frac{1}{2}$$
$$x=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$$

Memorize



Important Points on the Unit Circle

All students take calculus -mnemonic device

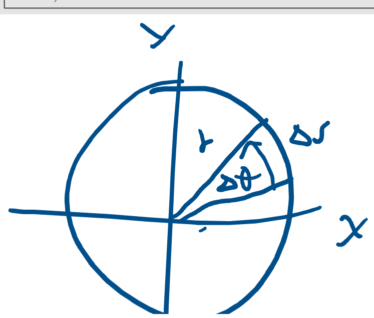
Memorize

Theorem 10.3. If $Q(x,y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2+y^2=r^2$ then $x=r\cos(\theta)$ and $y=r\sin(\theta)$. Moreover,

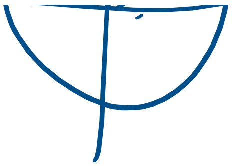
$$\cos(\theta)=\frac{x}{r}=\frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin(\theta)=\frac{y}{r}=\frac{y}{\sqrt{x^2+y^2}}$$

Memorize

Equation 10.3. Suppose an object is traveling in a circular path of radius r centered at the origin with constant angular velocity ω. If $t=0$ corresponds to the point $(r,0)$, then the x and y coordinates of the object are functions of t and are given by $x=r\cos(\omega t)$ and $y=r\sin(\omega t)$. Here, $\omega>0$ indicates a counter-clockwise direction and $\omega<0$ indicates a clockwise direction.



$$s=r\theta$$
$$\Delta s=r\Delta\theta$$
$$\frac{\Delta s}{\Delta t}=r\frac{\Delta\theta}{\Delta t}$$



x

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta \theta}{\Delta t}$$

$$V = r\omega$$

v = linear velocity

Memorize

Theorem 10.4. Suppose θ is an acute angle residing in a right triangle. If the length of the side adjacent to θ is a, the length of the side opposite θ is b, and the length of the hypotenuse is c, then $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$.

c = hypotenuse = hyp

b opposite opp

θ

a

adjacent

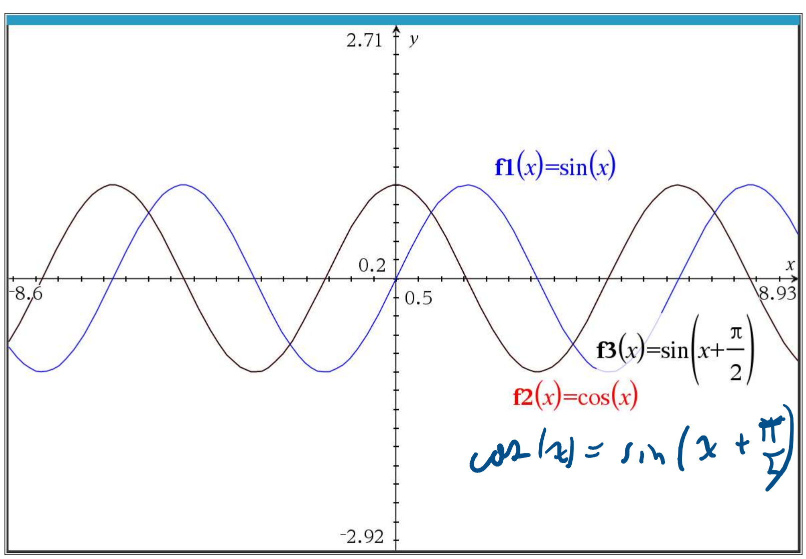
adj

$$\boxed{\cos(\theta) = \frac{adj}{hyp}} = \frac{a}{c}$$

$$\boxed{\sin(\theta) = \frac{opp}{hyp}} = \frac{b}{c}$$

Theorem 10.5. Domain and Range of the Cosine and Sine Functions:

- The function $f(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$



$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

10.3 memorize

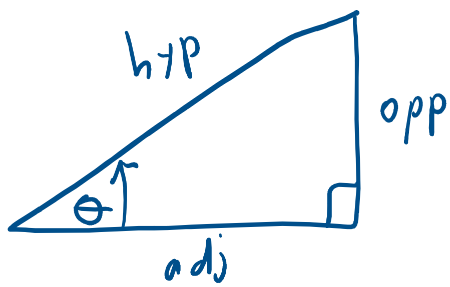
Definition 10.2. The Circular Functions: Suppose θ is an angle plotted in standard position and $P(x, y)$ is the point on the terminal side of θ which lies on the Unit Circle.

- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta) = x$.
- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta) = y$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \dfrac{1}{x}$, provided $x \neq 0$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \dfrac{1}{y}$, provided $y \neq 0$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

Memorize

Theorem 10.6. Reciprocal and Quotient Identities:

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.



memorize

$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp} = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj} = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp} = \frac{1}{\tan\theta}$$

soh cah toa

memorize

Theorem 10.8. The Pythagorean Identities:

1. $\cos^2(\theta) + \sin^2(\theta) = 1.$

   Common Alternate Forms:

   - $1 - \sin^2(\theta) = \cos^2(\theta)$

memorize

Theorem 10.8. The Pythagorean Identities:

1. $\cos^2(\theta) + \sin^2(\theta) = 1.$

   Common Alternate Forms:

   - $1 - \sin^2(\theta) = \cos^2(\theta)$
   - $1 - \cos^2(\theta) = \sin^2(\theta)$

2. $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

   Common Alternate Forms:

   - $\sec^2(\theta) - \tan^2(\theta) = 1$
   - $\sec^2(\theta) - 1 = \tan^2(\theta)$

3. $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

   Common Alternate Forms:

   - $\csc^2(\theta) - \cot^2(\theta) = 1$
   - $\csc^2(\theta) - 1 = \cot^2(\theta)$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$\tan^2\theta + 1 = \sec^2\theta$

$(a-b)(a+b) = a^2 - b^2$

Pythagorean Conjugates

- $1-\cos(\theta)$ and $1+\cos(\theta)$: $(1-\cos(\theta))(1+\cos(\theta)) = 1-\cos^2(\theta) = \sin^2(\theta)$
- $1-\sin(\theta)$ and $1+\sin(\theta)$: $(1-\sin(\theta))(1+\sin(\theta)) = 1-\sin^2(\theta) = \cos^2(\theta)$
- $\sec(\theta)-1$ and $\sec(\theta)+1$: $(\sec(\theta)-1)(\sec(\theta)+1) = \sec^2(\theta)-1 = \tan^2(\theta)$
- $\sec(\theta)-\tan(\theta)$ and $\sec(\theta)+\tan(\theta)$: $(\sec(\theta)-\tan(\theta))(\sec(\theta)+\tan(\theta)) = \sec^2(\theta)-\tan^2(\theta) = 1$
- $\csc(\theta)-1$ and $\csc(\theta)+1$: $(\csc(\theta)-1)(\csc(\theta)+1) = \csc^2(\theta)-1 = \cot^2(\theta)$
- $\csc(\theta)-\cot(\theta)$ and $\csc(\theta)+\cot(\theta)$: $(\csc(\theta)-\cot(\theta))(\csc(\theta)+\cot(\theta)) = \csc^2(\theta)-\cot^2(\theta) = 1$

Be able to apply

Strategies for Verifying Identities

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities in Theorem 10.6 to write functions on one side of the identity in terms of the functions on the other side of the identity. Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities in Theorem 10.8 to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities in Theorem 10.8.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.

10.3

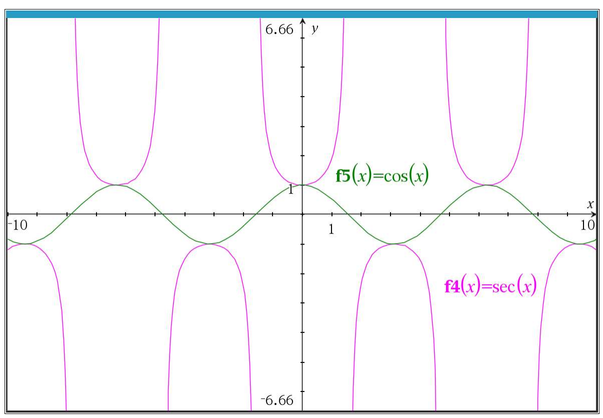
Theorem 10.11. Domains and Ranges of the Circular Functions

- The function $f(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$

memorize

- The function $F(t) = \sec(t) = \dfrac{1}{\cos(t)}$
  - has domain $\{t : t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\} = \bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$
  - has range $\{u : |u| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
- The function $G(t) = \csc(t) = \dfrac{1}{\sin(t)}$
  - has domain $\{t : t \neq \pi k, \text{for integers } k\} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $\{u : |u| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
- The function $J(t) = \tan(t) = \dfrac{\sin(t)}{\cos(t)}$
  - has domain $\{t : t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\} = \bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$
  - has range $(-\infty, \infty)$
- The function $K(t) = \cot(t) = \dfrac{\cos(t)}{\sin(t)}$
  - has domain $\{t : t \neq \pi k, \text{for integers } k\} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $(-\infty, \infty)$

supplied



10.4
Memorize

Theorem 10.12. Even / Odd Identities: For all applicable angles θ,

even
- $\cos(-\theta) = \cos(\theta)$
- $\sec(-\theta) = \sec(\theta)$

odd
- $\sin(-\theta) = -\sin(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

Def If $f(-x) = f(x)$ all $x \in$ domain of f then f is even

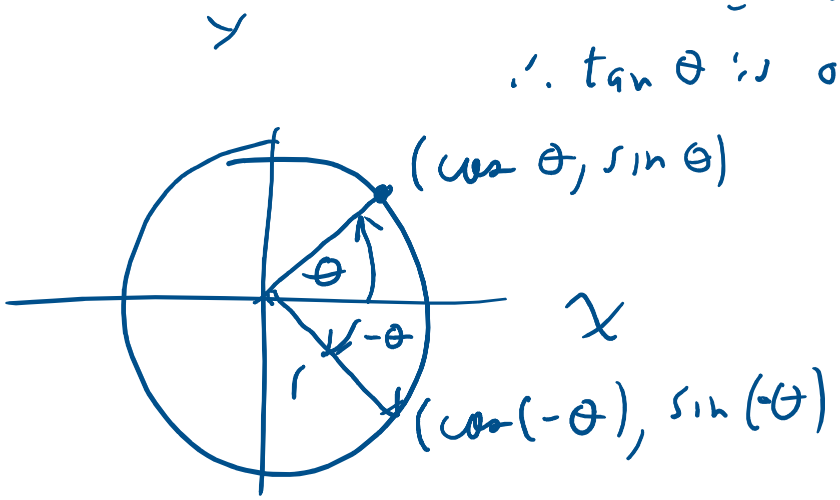
If $f(-x) = -f(x)$ all $x \in$ domain of f then f is odd

If $f(-x) = -f(x)$ all $x \in$ domain of f
then f is odd

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos\theta}$$

$$= -\tan\theta$$

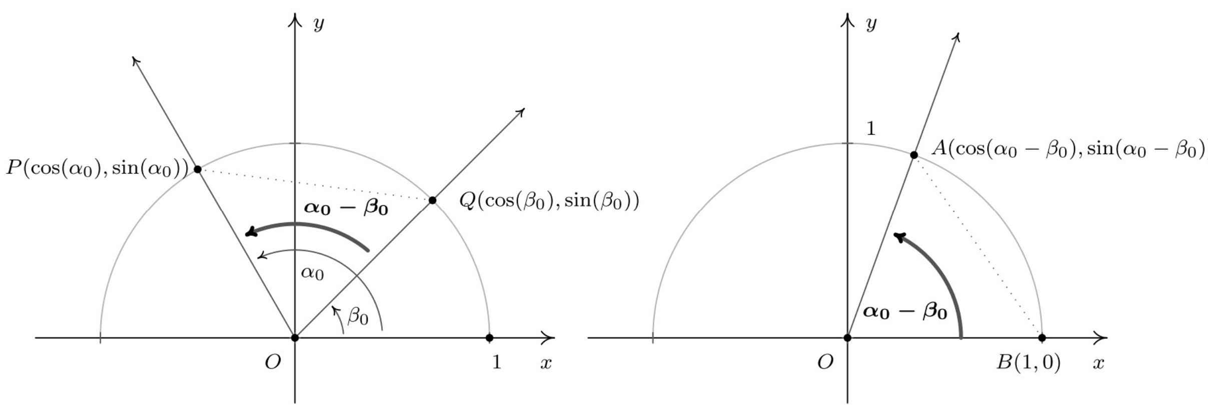
$\therefore \tan\theta$ is odd



Memorize

Theorem 10.13. Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$



Theorem 10.14. Cofunction Identities: For all applicable angles θ,

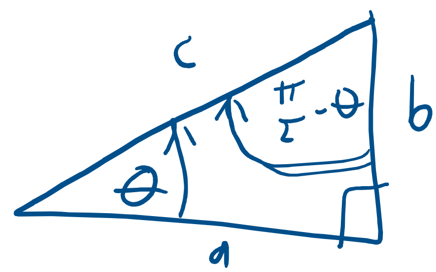
- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$
- $\sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta)$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$
- $\csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta)$
- $\cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)$

- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$
- $\csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta)$
- $\cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)$

$$\cos\left(\frac{\pi}{2} - \theta\right) \stackrel{?}{=} \sin\theta$$

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$$\cos\left(\frac{\pi}{2} - \theta\right)$$
$$= \cos\left(\frac{\pi}{2}\right)\cos(\theta) + \sin\left(\frac{\pi}{2}\right)\sin\theta$$
$$= (0)\cos\theta + (1)(\sin\theta)$$
$$= \sin\theta$$



$$\theta + w = \frac{\pi}{2}$$
$$w = \frac{\pi}{2} - \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{adj}{hyp} = \frac{b}{c}$$

$$\sin(\theta) = \frac{opp}{hyp} = \frac{b}{c}$$

$$\frac{b}{c} = \frac{b}{c}$$

$$\therefore \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

memorize

Theorem 10.15. Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

Theorem 10.16. Sum and Difference Identities: For all applicable angles α and β,

memorize
- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$

supplied
- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)(\sin\beta)$$

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$

Theorem 10.17. Double Angle Identities: For all applicable angles θ,

- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}$ memorize
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ memorize
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$ supplied

Given $\sin(a+b) = (\sin a)(\cos b) + (\cos a)(\sin b)$

derive $\sin(2\theta) = 2(\sin\theta)(\cos\theta)$

$$\sin(2\theta) = \sin(\theta+\theta)$$
$$= \sin\theta\cos\theta + \cos\theta\sin\theta$$
$$= 2\sin\theta\cos\theta$$

Memorize

Theorem 10.18. Power Reduction Formulas: For all angles θ,

- $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$
- $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$

Memorize

Theorem 10.19. Half Angle Formulas: For all applicable angles θ,

- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

Supplied

Theorem 10.20. Product to Sum Formulas: For all angles α and β,

- $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Supplied

Theorem 10.21. Sum to Product Formulas: For all angles α and β,

- $\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\cos(\alpha)-\cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
- $\sin(\alpha)\pm\sin(\beta) = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$

10.4

In Exercises 59 - 73, verify the identity. Assume all quantities are defined.

62. $\csc(2\theta) = \dfrac{\cot(\theta)+\tan(\theta)}{2}$

strategy
change to sin, cos

$\frac{1}{\sin(2\theta)}$

$$\frac{\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}}{2}$$

$$\frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin^2\theta}{\sin\theta\cos\theta}$$

$$\frac{\frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin^2\theta}{\sin\theta\cos\theta}}{2}$$

$$\frac{\frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin^2\theta}{\sin\theta\cos\theta}}{\frac{2}{1}}$$

$$\frac{\cos^2\theta + \sin^2\theta}{2\sin\theta\cos\theta}$$

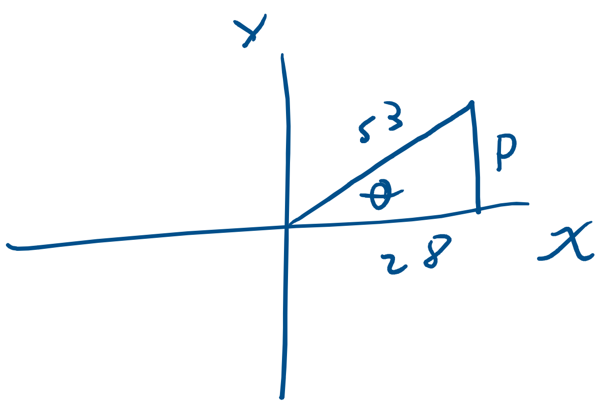
$$\frac{1}{\sin(2\theta)} = \frac{1}{\sin(2\theta)} \quad \checkmark$$

10.4

In Exercises 49 - 58, use the given information about θ to find the exact values of

- $\sin(2\theta)$
- $\cos(2\theta)$
- $\tan(2\theta)$
- $\sin\left(\dfrac{\theta}{2}\right)$
- $\cos\left(\dfrac{\theta}{2}\right)$
- $\tan\left(\dfrac{\theta}{2}\right)$

50. $\cos(\theta) = \dfrac{28}{53}$ where $0 < \theta < \dfrac{\pi}{2}$



Find $\sin(2\theta)$

$\sin\theta = \dfrac{opp}{} = \dfrac{p}{}$

$\sin\theta = \frac{opp}{hyp} = \frac{p}{53}$

$$\boxed{p = \sqrt{53^2 - 28^2}}$$

$\sin(2\theta) = 2\sin\theta\cos\theta$