09-30-25 MTH 167-004N

3 Polynomial Functions

3.1 Graphs of Polynomials
3.1.1 Exercises
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3.2 The Factor Theorem and the Remainder Theorem
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3.3 Real Zeros of Polynomials
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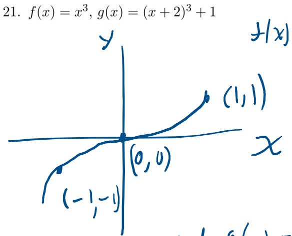
3.1: 21

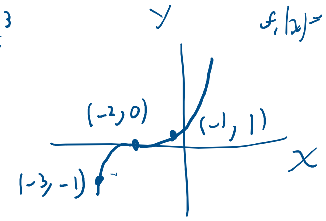
In Exercises 21 - 26, given the pair of functions f and g, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice through the transformations. State the domain and range of g.

21. $f(x) = x^3$, $g(x) = (x+2)^3 + 1$

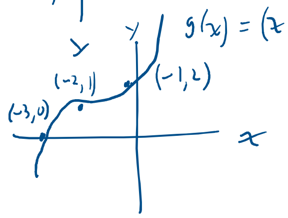
$f(x) = x^3$



$f_1(x) = (x+2)^3$



$g(x) = (x+2)^3 + 1$



domain of $g = (-\infty, \infty)$

3.2
Supplied

Theorem 3.4. Polynomial Division: Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

$p(x) =$ polynomial
$d(x) =$ divisor
$q(x) =$ quotient
$r(x) =$ remainder

Supplied

Theorem 3.5. The Remainder Theorem: Suppose p is a polynomial of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$.

Thm. 3.4 $\quad p(x) = d(x)\,q(x) + r(x)$
$d(x) = x - c$
$\Rightarrow p(x) = (x-c)\,q(x) + r(x)$
show $r(x) = p(c)$
$P(c) = (c - c)\,q(c) + r(c)$
$p(c) = 0 \cdot q(c) + r(c)$
$\boxed{P(c) = r(c)}$

What about r(x) for any x?

$r(x) = 0$ or $\deg r(x) < \deg(x - c) = 1$
$r(x) = 0$ or $\deg r(x) = 0$

If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.

$\deg r(x) = 0 \Rightarrow r(x) = a_0 \neq 0$
constant

$\Rightarrow r(x) = r(c) = a_0$ for any x
$\therefore p(c) = r(x)$

$r(x) = 0$ for all x

$\therefore p(c) = r(x)$

$r(x) = 0$ for all x

$\therefore p(c) = r(c) = 0$

Memorize

Theorem 3.6. The Factor Theorem: Suppose p is a nonzero polynomial. The real number c is a zero of p if and only if $(x-c)$ is a factor of $p(x)$.

Assume $x-c$ is a factor of $P(x)$

Prove $x = c$ is a zero of $p(x)$

Thm 3.4 $p(x) = d(x)\,q(x) + r(x)$

$p(x) = (x-c)\,q(x) + r(x)$

$r(x) = 0$

$P(x) = (x-c)\,q(x)$

show $x=c$ is a zero of $p(x)$

i.e. show $p(c) = 0$

$p(c) = (c-c)\,q(c)$

$p(c) = 0 \cdot q(c)$

$p(c) = 0$

Memorize

Theorem 3.7. Suppose f is a polynomial of degree $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

$f(x) = 9x - 5 = 0$

$9x = 5$

$\boxed{x = \frac{5}{9}}$

deg of $f = 1$

$f(x) = (x-2)^2(x^2+1)$

deg of $f = 4$

zeros 2

mult 2

2 real zeros, counting multiplicity

$2 \leq 4$

Memorize

Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x-c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

Long division of polynomials

$(x^3 + 4x - 10) \div (x+6)$

$x^2 - 6x + 40 - \frac{250}{x+6}$

$$x+6 \overline{\big) \, x^3 + 0 \cdot x^2 + 4x - 10} \quad x^2 - 6x + 40 - \frac{250}{x+6}$$

$$\begin{array}{r}
x^3 + 6x^2 \\
\hline
-6x^2 + 4x \\
-6x^2 - 36x \\
\hline
40x - 10 \\
40x + 240 \\
\hline
-250
\end{array}$$

$x^3 - x^3 = 0$

$-6x^2 - (-6x^2)$
$= -6x^2 + 6x^2 = 0$

Check: $(x+6)\left(x^2 - 6x + 40 - \frac{250}{x+6}\right)$

$$= x^3 - 6x^2 + 40x$$
$$6x^2 - 36x + 240$$
$$+ (x+6)\left(-\frac{250}{x+6}\right)$$

$$= x^3 + 4x + 240 - 250$$
$$= x^3 + 4x - 10 \quad \checkmark$$

$$p(x) = d(x)\,q(x) + r(x)$$

$$x^3 + 4x - 10 = (x+6)(x^2 - 6x + 40) - 250$$

$$\frac{x^3 + 4x - 10}{x+6} = x^2 - 6x + 40 - \frac{250}{x+6}$$

Synthetic division only applies with linear divisors

$$(x^3 + 4x - 10) \div (x+6)$$

$$\begin{array}{r|rrrr}
-6 & 1 & 0 & 4 & -10 \\
 & & -6 & 36 & -240 \\
\hline
 & 1 & -6 & 40 & (-250)
\end{array}$$

mult

remainder

$$x^2 - 6x + 40 - \frac{250}{x+6}$$

3.2

In Exercises 1 - 6, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

4. $(-x^5 + 7x^3 - x) \div (x^3 - x^2 + 1)$

$$x^3 - x^2 + 1 \overline{\big) -x^5 + 0x^4 + 7x^3 + 0x^2 - x + 0} \quad -x^2 - x + 6 + \frac{7x^2 - 6}{x^3 - x^2 + 1}$$

$$\begin{array}{r}
-x^5 + x^4 + 0x^3 - x^2 + 0x \\
\hline
-x^4 \quad 7x^3 \quad x^2 \quad -x \\
-x^4 \quad +x^3 \quad 0x^2 \quad -x \\
\hline
6x^3 \quad +x^2 \qquad 0
\end{array}$$

$$\begin{array}{rrrr} -x\cdot & 7x^2 & x^2 & -x \\ -x^4 & +x^3 & 0x^2 & -x \\ \hline & 6x^3 & +x^2 & 0 \\ & 6x^3 & -6x^2 & +6 \\ \hline & & 7x^2 & -6 \end{array}$$

$$-x^5+7x^3-x=(x^3-x^2+1)(-x^2-x+6)+7x^2-6$$

3.2

In Exercises 31 - 40, you are given a polynomial and one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

34. $2x^3-3x^2-11x+6, \quad c=\frac{1}{2}$

$2x^3-3x^2-11x+6=\left(x-\frac{1}{2}\right)q(x)$

Use synthetic division to find $q(x)$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -3 & -11 & 6 \\ & & 1 & -1 & -6 \\ \hline & 2 & -2 & -12 & (0) \end{array}$$

$q(x)=2x^2-2x-12$

solve $q(x)=0$

$2x^2-2x-12=0$

$x^2-x-6=0$

$(x+2)(x-3)=0$

$\boxed{x=-2,3}$

3.2

In Exercises 41 - 45, create a polynomial p which has the desired characteristics. You may leave the polynomial in factored form.

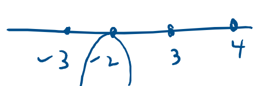
44.
- The solutions to $p(x)=0$ are $x=\pm3$, $x=-2$, and $x=4$.
- The leading term of $p(x)$ is $-x^5$.
- The point $(-2,0)$ is a local maximum on the graph of $y=p(x)$.

$p(x)=a(x-3)^?(x+3)^?(x+2)^?(x-4)^?$

$a=$ constant <0

1 multiplicity must be 2

$\Rightarrow -x^5=a\cdot x^5 \Rightarrow \boxed{a=-1}$



mult even $\rightarrow$ mult of $x=-2$ is 2

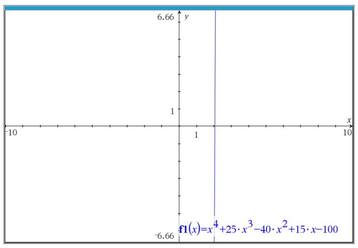
$$\boxed{p(x)=-(x-3)(x+3)(x+2)^2(x-4)}$$

$$\boxed{p(x) = -(x-3)(x+3)(x+2)^2(x-4)}$$

3.3 supplied

Theorem 3.8. Cauchy's Bound: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}$. Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.

What is a practical application of Cauchy's Bound Theorem to our MTH 167 class?
We can use this theorem to set our calculator window to include all real zeros.

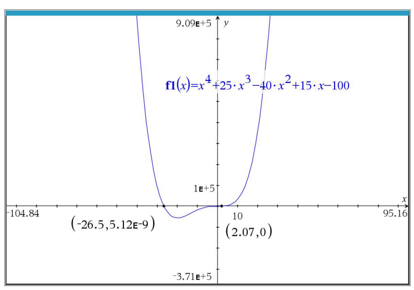


$a_n = a_4 = 1$
$a_3 = 25$
$a_2 = -40$
$a_1 = 15$
$a_0 = -100$

$M = \max\{|25|, |-40|, |15|, |-100|\}$
$= \max\{25, 40, 15, 100\}$
$= 100$

$\Rightarrow M+1 = 100$

$\Rightarrow$ all real zero are $(-100, 100)$



solve$(x^4+25\cdot x^3-40\cdot x^2+15\cdot x-100=0,x)$ $\quad x=-26.5341$ or $x=2.07117$

cSolve$(x^4+25\cdot x^3-40\cdot x^2+15\cdot x-100=0,x)$
$x=-0.268512+1.32194\cdot i$ or $x=-0.268512-1.32194\cdot i$ or $x=-26.5341$ or $x=2.07117$

Supplied

Theorem 3.9. Rational Zeros Theorem: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and $a_0, a_1, \ldots a_n$ are integers. If r is a rational zero of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

$f(x) = (3x-1)(x+4)(x^2-2)$

$= (3x^2 + 12x - x - 4)(x^2-2)$

$= 3x^4 - 6x^2 + 12x^3 - 24x - x^3 + 2x - 4x^2 + 8$

$f(x) = 3x^4 + 11x^3 - 10x^2 - 22x + 8$

$a|b$ means a divides b, or a is a factor of b

$a|b$ means a divides b, or a is a factor of b

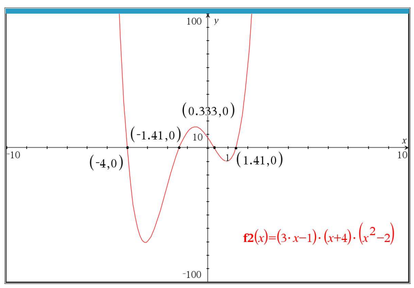
If $r = \frac{p}{q}$ is a rational zero of $f(x)$

Then $p|8$ and $q|3$

$p = \pm 1, \pm 2, \pm 4, \pm 8$

$q = \pm 1, \pm 3$

$\frac{p}{q} = \pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}, \pm 4, \pm \frac{4}{3}, \pm 8, \pm \frac{8}{3}$



From graph

$x \approx -4, -1.41, 0.333, 1.41$

$x = -4, \frac{1}{3}$ are rational zeros

$x = \pm 1.41$ are not in the list of possible rational zeros

$\therefore$ they are irrational

Gemini

2.5 Flash

Upgrade

Can you explain why synthetic division of polynomials works?

Show thinking

Synthetic division is a **shorthand method** for polynomial long division, designed specifically for the case when the divisor is a **linear binomial** of the form $(x - c)$. Its effectiveness stems from the fact that it automates and simplifies the repetitive steps in long division by focusing only on the coefficients.

Here's a breakdown of why it works:

1. Coefficient Only

Polynomial long division involves a lot of writing out variables and powers of x. Since you are dividing by a linear term, the power of x in each term of the quotient is always one less than the corresponding power in the dividend.

- Synthetic division eliminates the need to write the variables (x^n, x^{n-1}, etc.), focusing only on the **coefficients** of the terms in the polynomial. The position of each number automatically indicates the power of x it corresponds to.

2. Simplified Subtraction

In polynomial long division, a key step is multiplying a term in the quotient by the divisor and then **subtracting** the result from the dividend. This subtraction often leads to sign errors.

- Synthetic division uses the **root** (c) of the divisor $(x - c)$, instead of the divisor itself. If you set $x - c = 0$, you get $x = c$.
- By using c (instead of $-c$) for the multiplication step, the process naturally incorporates the necessary sign change, allowing all subsequent operations to be **addition** instead of subtraction. This simplifies the arithmetic and reduces errors.

3. Connection to the Remainder Theorem

The process of synthetic division is also mathematically equivalent to **Horner's method** for evaluating a polynomial, which is closely tied to the **Remainder Theorem**.

- The **Division Algorithm** for polynomials states that when a polynomial $P(x)$ is divided by a divisor $D(x)$, we get a quotient $Q(x)$ and a remainder $R(x)$ such that:

$$P(x) = D(x) \cdot Q(x) + R(x)$$

- If the divisor is $D(x) = x - c$, this becomes:

$$P(x) = (x - c) \cdot Q(x) + R$$

(The remainder R is a constant because the divisor is degree 1).

- The **Remainder Theorem** states that $P(c) = R$. If you plug c into the equation above:

$$P(c) = (c - c) \cdot Q(c) + R$$

$$P(c) = (0) \cdot Q(c) + R$$

$$P(c) = R$$

- Synthetic division, by using c and the steps of multiply-and-add, essentially **calculates** $P(c)$. Therefore, the final number (the remainder) generated by the synthetic division process is the value of the polynomial at $x = c$, and the preceding numbers are the coefficients of the quotient. If the remainder is zero, the **Factor Theorem** tells us that $(x - c)$ is a factor of the polynomial.

The fundamental reason synthetic division works is that it is a **condensed and optimized version of polynomial long division** when the divisor is a monic linear factor.

You can learn more about this by watching
[Why synthetic division works].