10-21-25 MTH 167-004N

5 Further Topics in Functions
5.1 Function Composition
 5.1.1 Exercises
 page 369 (381): 1, 9, 13, 17, 22, 31, 39, 62

Exam 2, Thursday, 10/23/25, 2.2 - 2.4, 3.1 - 3.4, 4.1 - 4.3, 5.1

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14 class meetings before final exam
24 required textbook sections
24/14=1.7143 sections per class meeting

After class notes

Gemini
2.5 Flash

Give me a rational inequality without solving it.

A rational inequality is an inequality that contains a rational expression (a fraction where the numerator and denominator are polynomials).

Here is an example of a rational inequality:

$$\frac{x^2-4}{x+1} \leq 0$$

Components of the Inequality

- **Rational Expression:** $\frac{x^2-4}{x+1}$
- **Numerator:** x^2-4
- **Denominator:** $x+1$
- **Inequality Sign:** $\leq$ (less than or equal to)
- **Right-Hand Side:** 0

Let $f(x) = \frac{x^2-4}{x+1}$

The numerator of the rational inequality $\frac{x^2-4}{x+1} \leq 0$ is x^2-4.

solve $f(x) \leq 0$

Factoring the numerator:

The expression x^2-4 is a **difference of squares**, which follows the pattern $a^2-b^2=(a-b)(a+b)$.

$$\mathbf{x^2-4=(x-2)(x+2)}$$

$$\frac{(x-2)(x+2)}{x+1} \leq 0$$

$x+1=0 \Rightarrow x=-1$ which is not in the domain

$x + 1$

$x+1=0 \Rightarrow x=-1$ which is not in the domain

How do I know if x = -1 is a hole or a vertical asymptote?

To determine if the discontinuity at $x=-1$ is a **hole** (removable discontinuity) or a **vertical asymptote** (non-removable discontinuity), you must **factor** the numerator and denominator of the rational function and look for a common factor of $(x+1)$.

Here is the step-by-step method and the condition for each type of discontinuity:

1. Factor the Function

Start with your rational function, $f(x)=\frac{N(x)}{D(x)}$, and factor both the numerator, $N(x)$, and the denominator, $D(x)$, completely.

Since the discontinuity is at $x=-1$, you are looking for the factor $(x+1)$ in the denominator, $D(x)$.

2. Check for Cancellation

The nature of the discontinuity at $x=-1$ depends entirely on whether the factor $(x+1)$ also appears in the numerator, $N(x)$, and can be canceled out.

If it's a Hole (Removable Discontinuity)

A **hole** exists at $x=-1$ if and only if the factor $(x+1)$ **cancels out** of the denominator when you simplify the rational function.

$$\text{Example: } f(x)=\frac{(x+1)(x-2)}{(x+1)(x+3)}$$

- The factor $(x+1)$ appears in both the numerator and the denominator, so it can be canceled.
- The function is undefined at $x=-1$ (because the original denominator is zero), but the simplified function would be $g(x)=\frac{x-2}{x+3}$ for $x\neq-1$.
- The graph behaves normally near $x=-1$, but there is a **single point of discontinuity** (a hole) at $(-1,L)$, where L is the value of the simplified function at $x=-1$.

If it's a Vertical Asymptote (Non-removable Discontinuity)

A **vertical asymptote** exists at $x=-1$ if the factor $(x+1)$ **remains** in the denominator after simplifying the rational function.

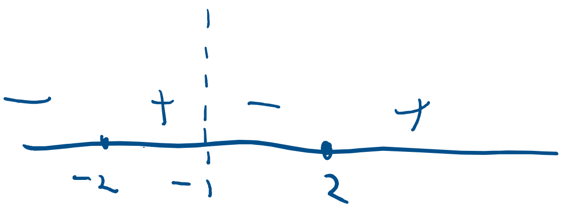
$$\text{Example: } f(x)=\frac{x-2}{(x+1)(x+3)}$$

- The factor $(x+1)$ is **only** in the denominator and **cannot** be canceled.
- This means as x approaches -1, the denominator approaches 0 while the numerator approaches a non-zero number, causing the function's value to approach $\pm\infty$.
- The graph will have a vertical line that the function approaches but never touches.

Summary of the Condition

Condition (After Factoring)	Result at $\mathbf{x} = -1$	Graphical Behavior
$(x+1)$ **cancels** out of the denominator.	**Hole** (Removable Discontinuity)	The graph has a single open circle.
$(x+1)$ **remains** in the denominator.	**Vertical Asymptote** (Non-removable)	The function shoots up/down to $\pm\infty$.

$$f(x) = \frac{x^2-4}{x+1} = \frac{(x+2)(x-2)}{x+1}$$



$$f(-3) = \frac{(-3+2)(-3-2)}{-3+1} = \frac{(-)(-)}{(-)} < 0$$

$$f\left(-\tfrac{3}{2}\right) = \frac{\left(-\frac{3}{2}+2\right)\left(-\frac{3}{2}-2\right)}{-\frac{3}{2}+1} = \frac{(+)(-)}{(-)} > 0$$

$$f(0) = \frac{(0+2)(0-2)}{0+1} = \frac{(+)(-)}{(+)} < 0$$

$$f(3) = \frac{(3+2)(3-2)}{3+1} = \frac{(+)(+)}{+} > 0$$

solution set

$$(-\infty,-2]\cup(-1,2]$$

5.1: 39

In Exercises 31 - 40, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

$$39.\quad v(x) = \frac{2x+1}{3-4x}$$

$$v(x) = (v\circ I)(x) = v(I(x)) = v(x)$$

$$v(x) = \frac{2x+1}{3-2(2x)}$$

Let $g(x) = 2x$

$$f(x) = \frac{x+1}{3-2x}$$

$(f\circ g)(x) = f(2x)$

$3 - 2x$

$(f \circ g)(x) = f(2x)$

$= \frac{2x+1}{3-2(2x)}$

$= \frac{2x+1}{3-4x} = v(x)$

5.1:1

In Exercises 1 - 12, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

1. $f(x) = x^2$, $g(x) = 2x + 1$

$(g \circ f)(0) = g(f(0))$

$= g(0^2)$

$= g(0)$

$= 2(0)+1$

$= \boxed{1}$

$(g \circ f)(x) = g(f(x))$

$= g(x^2)$

$\boxed{(g \circ f)(x) = 2x^2+1}$

$(g \circ f)(0) = 2(0^2)+1 = \boxed{1}$

$(f \circ g)\left(\frac{1}{2}\right)$

$= f\left(g\left(\frac{1}{2}\right)\right)$

$= f\left(2\left(\frac{1}{2}\right)+1\right)$

$= f(1+1)$

$= f(2)$

$= 2^2 = \boxed{4}$

5.1: 62

62. The volume V of a cube is a function of its side length x. Let's assume that $x = t + 1$ is also a function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.

Find a formula for $V(t)$

Find a formula for $V(t)$

$$\boxed{V(x) = x^3}$$

$$\boxed{x = t+1}$$

$$\boxed{V(t) = (t+1)^3}$$

2.2: 17

Prove that if $|f(x)| = |g(x)|$ then either $f(x) = g(x)$ or $f(x) = -g(x)$. Use that result to solve the equations in Exercises 16 - 21.

17. $|3x+1| = |4x|$

$3x+1 = 4x$ or $3x+1 = -4x$

$\boxed{x=1}$

$7x = -1$

$\boxed{x = -\frac{1}{7}}$

Given $|f(x)| = |g(x)|$

Prove $f(x) = g(x)$ or $f(x) = -g(x)$

Def $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Case 1) $f(x) \geq 0$ and $g(x) \geq 0$

$\Rightarrow |f(x)| = f(x)$

$|g(x)| = g(x)$

$\Rightarrow f(x) = g(x)$

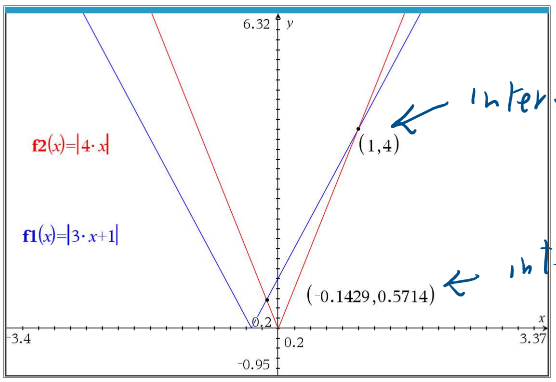
Case 2) $f(x) \geq 0$ and $g(x) < 0$

$|f(x)| = f(x)$

Case 2) ...

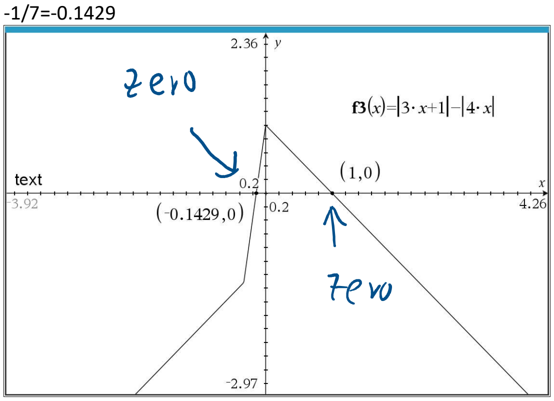
$$|f(x)| = f(x)$$
$$|g(x)| = -g(x)$$
$$\Rightarrow f(x) = -g(x)$$

Other cases similar.



$x \approx 1, -0.1429$

-1/7=-0.1429



$x \approx 1, -0.1429$

Motivation for completing the square.

1. Let $f(x) = x^2 + 5x + 2$.

$$p^2 + 2pq + q^2 = (p+q)^2$$

$$p \leftrightarrow x$$

$$x^2 + 2qx + q^2$$

$$x^2 + 5x + 2$$

$$2q = 5$$

$$\Rightarrow q = \frac{5}{2} \Rightarrow q^2 = \frac{25}{4}$$

$$x^2 + 5x + \underbrace{\frac{25}{4} - \frac{25}{4}}_{0} + 2$$

$$\left(x^2 + 5x + \frac{25}{4}\right) - \frac{25}{4} + 2$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{8}{4}$$

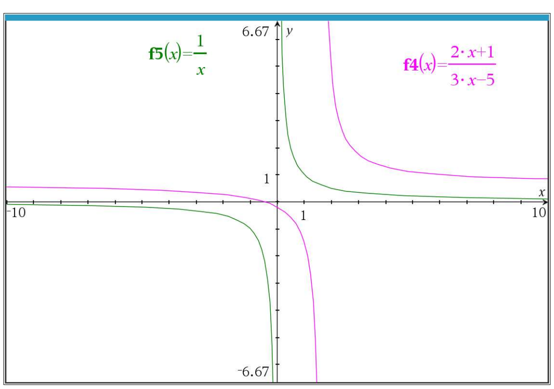
$$\boxed{\left(x + \frac{5}{2}\right)^2 - \frac{17}{4}}$$

$$\left(x - \left(-\frac{5}{2}\right)\right)^2 - \frac{17}{4}$$

vertex $\left(-\frac{5}{2}, -\frac{17}{4}\right)$

4.2: 21

21. Discuss with your classmates how you would graph $f(x) = \dfrac{ax+b}{cx+d}$. What restrictions must be placed on a, b, c and d so that the graph is indeed a transformation of $y = \dfrac{1}{x}$?



What happens if c = 0?
We get a line.

a = c and b = d, then f(x) = 1, horizontal line

$$x^2+3 \overline{\big)\, 5x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1}$$

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Final Result

The result of the polynomial long division is:

Quotient

$$\mathbf{Q(x) = 5x^3 - 4x^2 - 12x + 10}$$

Remainder

$$\mathbf{R(x) = 37x - 31}$$

This means the original expression can be written as:

$$5x^3 - 4x^2 - 12x + 10 + \frac{37x - 31}{x^2 + 3}$$

3.2: 42

In Exercises 41 - 45, create a polynomial p which has the desired characteristics. You may leave the polynomial in factored form.

42.
- The zeros of p are $c = 1$ and $c = 3$
- $c = 3$ is a zero of multiplicity 2.
- The leading term of $p(x)$ is $-5x^3$

$$\left.\begin{aligned} p_1(x) &= (x-1)(x-3) \\ p_2(x) &= 2(x-1)(x-3) \\ p_3(x) &= 2(x-1)^2(x-3)^4 \end{aligned}\right\}$$ if only the first property applies

$$p(x) = a(x-1)^m (x-3)^n$$

$$p(x) = a(x-1)^m (x-3)^2$$

$m = 1$

$$p(x) = a(x-1)(x-3)^2$$

leading term $= ax^3 = -5x^3$

$\Rightarrow \boxed{a = -5}$

$$\boxed{p(x) = -5(x-1)(x-3)^2}$$