12-04-25 MTH 167-002N

8 Systems of Equations and Matrices
8.1 Systems of Linear Equations: Gaussian Elimination
8.1.1 Exercises
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8.2 Systems of Linear Equations: Augmented Matrices
8.2.1 Exercises
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Omit vectors; omit 11.8, 11.9

8.1 memorize

Definition 8.1. A **linear equation in two variables** is an equation of the form $a_1x + a_2y = c$ where a_1, a_2 and c are real numbers and at least one of a_1 and a_2 is nonzero.

Memorize

Definition 8.2. A **linear equation in n variables**, $x_1, x_2, \ldots, x_n$, is an equation of the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ where $a_1, a_2, \ldots a_n$ and c are real numbers and at least one of a_1, $a_2, \ldots, a_n$ is nonzero.

Theorem 8.1. Given a system of equations, the following moves will result in an equivalent system of equations.

- Interchange the position of any two equations.
- Replace an equation with a nonzero multiple of itself.[a]
- Replace an equation with itself plus a nonzero multiple of another equation.

[a]That is, an equation which results from multiplying both sides of the equation by the same nonzero number.

8.2
Memorize

Theorem 8.2. Row Operations: Given an augmented matrix for a system of linear equations, the following row operations produce an augmented matrix which corresponds to an equivalent system of linear equations.

- Interchange any two rows.
- Replace a row with a nonzero multiple of itself.[a]
- Replace a row with itself plus a nonzero multiple of another row.[b]

[a]That is, the row obtained by multiplying each entry in the row by the same nonzero number.
[b]Where we add entries in corresponding columns.

Be able to recognize rref

Definition 8.4. A matrix is said to be in **row echelon form** provided all of the following conditions hold:

1. The first nonzero entry in each row is 1.
2. The leading 1 of a given row must be to the right of the leading 1 of the row above it.
3. Any row of all zeros cannot be placed above a row with nonzero entries.

Definition 8.5. A matrix is said to be in **reduced row echelon form** provided both of the following conditions hold:

rref

1. The matrix is in row echelon form.
2. The leading 1s are the only nonzero entry in their respective columns.

8.2

In Exercises 7 - 12, the following matrices are in reduced row echelon form. Determine the solution of the corresponding system of linear equations or state that the system is inconsistent.

8. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 19 \end{array}\right]$ rref

$1\cdot x + 0\cdot y + 0 z = -3$

$0\cdot x + 1\cdot y + 0\cdot z = 20$

$0\cdot x + 0\cdot y + 1\cdot z = 19$

$$\boxed{\begin{aligned} x &= -3 \\ y &= 20 \\ z &= 19 \end{aligned}}$$

9. $\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 6 & -6 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right] \end{array}$

$x_1 + 3x_4 = 4$

$x_2 + 6x_4 = -6$

$x_1 + 3x_4 = 1$

$x_2 + 6x_4 = -6$

$x_3 = 2$

$$\boxed{\begin{aligned} x_1 &= 4 - 3x_4 \\ x_2 &= -6 - 6x_4 \\ x_3 &= 2 \\ x_4 &= \text{free variable, parametric} \end{aligned}}$$

10. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

$x_1 = -3x_4$

$x_2 = -2x_3 - 6x_4$

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 1$

$0 = 1$ false

$\therefore$ no solution

The system is inconsistent

Solve the system of linear equations by transforming the augmented matrix to rref. Then, check with calculator.

14. $\begin{cases} x + y + z = 3 \\ 2x - y + z = 0 \\ -3x + 5y + 7z = 7 \end{cases}$

augmented matrix

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \end{array}\right.$

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 1 & 0 \\ -3 & 5 & 7 & 7 \end{array}\right] \begin{array}{l} R2-2R1 \\ R3+3R1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2-2(1) & -1-2(1) & 1-2(1) & 0-2(3) \\ -3+3(1) & 5+3(1) & 7+3(1) & 7+3(3) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & -1 & -6 \\ 0 & 8 & 10 & 16 \end{array}\right] \begin{array}{l} -R2 \\ R2/2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 6 \\ 0 & 4 & 5 & 8 \end{array}\right] \begin{array}{l} 4R2 \\ 3R3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 12 & 4 & 24 \\ 0 & 12 & 15 & 24 \end{array}\right] R3-R2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 12 & 4 & 24 \\ 0 & 0 & 11 & 0 \end{array}\right] \begin{array}{l} R2/4 \\ R3/11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \end{array}\right. R1-R3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R1 - R3 \\ R2 - R3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & 0 \end{array}\right] R2/3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] R1 - R2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{rref}$$

$$\boxed{\begin{array}{l} x = 1 \\ y = 2 \\ z = 0 \end{array}}$$

15. $\begin{cases} 4x - y + z = 5 \\ 2y + 6z = 30 \\ x + z = 5 \end{cases}$

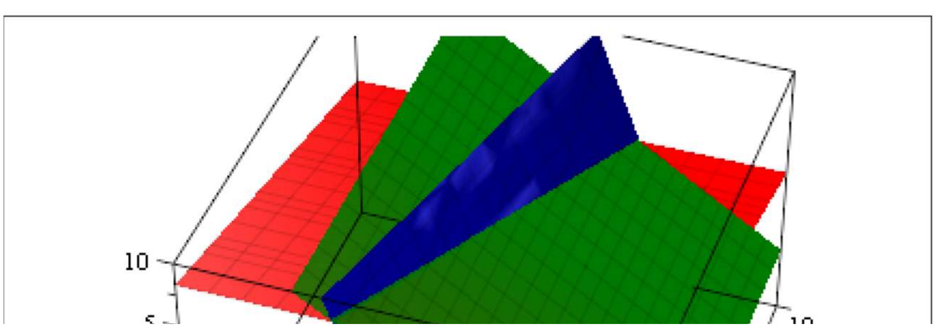
$$\left[\begin{array}{ccc|c} 4 & -1 & 1 & 5 \\ 0 & 2 & 6 & 30 \\ 1 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} R1 - 4R3 \\ R2/2 \\ \end{array}$$

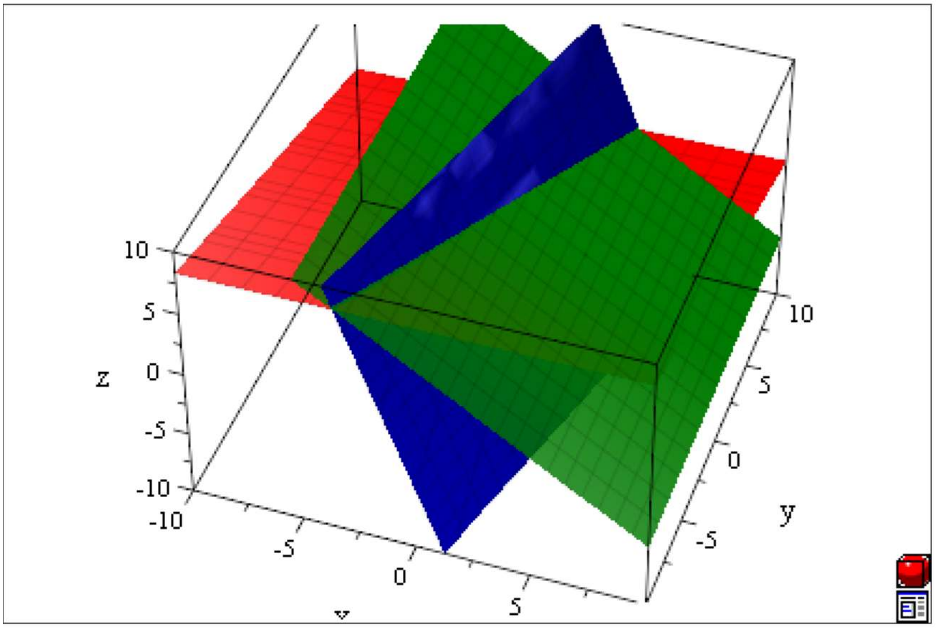
$$\left[\begin{array}{ccc|c} 0 & -1 & -3 & -15 \\ 0 & 1 & 3 & 15 \\ 1 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} R1 + R2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 15 \\ 1 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} \text{interchange} \\ R1,\ R3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 3 & 15 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ rref}$$

$$\boxed{\begin{array}{l} x = 5 - z \\ y = 15 - 3z \\ z = \text{free variable} \end{array}}$$





The three planes intersect in a line in space.

Copilot

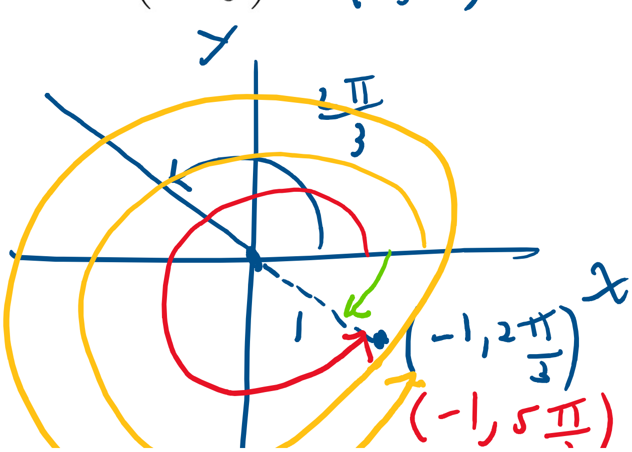
With systems that once seemed a chore,
We line up the rows to explore.
Swap, scale, and combine,
Step by step they align,
Till RREF opens the door!

11.4: 11

11.4.1 Exercises

In Exercises 1 - 16, plot the point given in polar coordinates and then give three different expressions for the point such that (a) $r < 0$ and $0 \leq \theta \leq 2\pi$, (b) $r > 0$ and $\theta \leq 0$ (c) $r > 0$ and $\theta \geq 2\pi$

11. $\left(-1, \frac{2\pi}{3}\right) = (r, \theta)$



(a) $\theta = \frac{2\pi}{3} + \pi$

$\boxed{\theta = \frac{5\pi}{3}}$

(b) $\left(1, -\frac{\pi}{3}\right)$

(c) $\frac{5\pi}{3} + 2\pi$

$= 11\pi$

$\frac{}{3})$

$(-1, 5\frac{\pi}{3})$

$(1, 11\frac{\pi}{3})$

$= \frac{11\pi}{3}$

Find the rectangular coordinates of this point

$x = r\cos\theta$

$y = r\sin\theta$

$(r, \theta) = (1, -\frac{\pi}{3})$

$x = \cos(-\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$

$y = \sin(-\frac{\pi}{3}) = -\sin(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

$\boxed{(x, y) = (\frac{1}{2}, -\frac{\sqrt{3}}{2})}$

write in polar coordinates

$x^2 + y^2 = r^2$

$(\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2 = r^2$

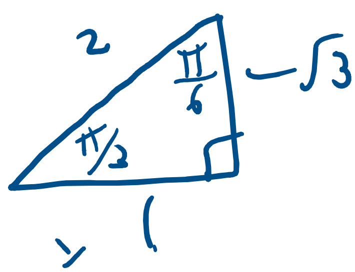
$\frac{1}{4} + \frac{3}{4} = r^2$

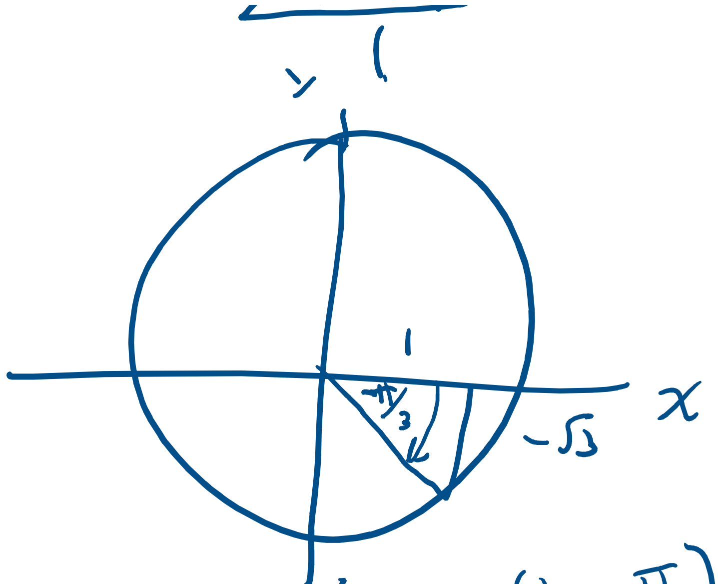
$1 = r^2$

$\boxed{r = \pm 1}$

$\theta = \arctan(\frac{y}{x}) = \arctan\left(\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)$

$= \arctan(-\sqrt{3})$





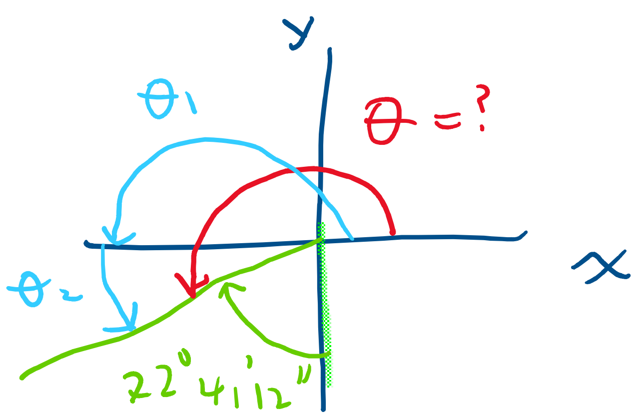
$(r, \theta) = (1, -\frac{\pi}{3})$

11.2: 25

25. Find the angle θ in standard position with $0° \le \theta < 360°$ which corresponds to each of the bearings given below.

(a) due west (b) S83°E (c) N5.5°E (d) due south

(e) N31.25°W (f) S72°41′12″W[15] (g) N45°E (h) S45°W



$\theta = \theta_1 + \theta_2$

$\boxed{\theta_1 = 180°}$

$\theta_2 = 90° - (72°41'12'')$

$$\theta_2 = 90 - \left(72 + \frac{41}{60} + \frac{12}{3600}\right)$$

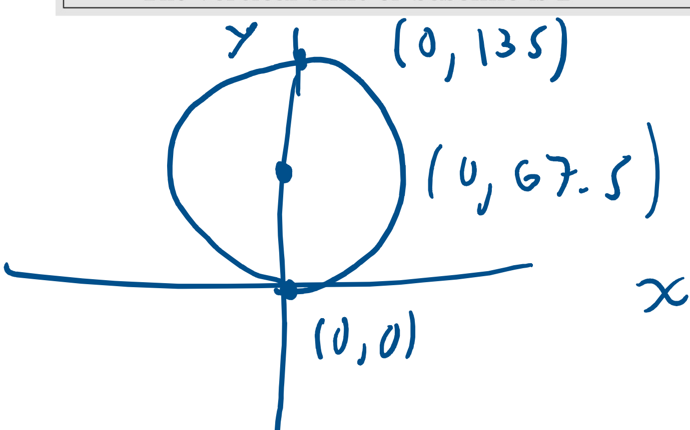
$$\theta = 197.\overline{333}.$$

Ans DMS $= 197°18'48''$

3. The London Eye is a popular tourist attraction in London, England and is one of the largest Ferris Wheels in the world. It has a diameter of 135 meters and makes one revolution (counter-clockwise) every 30 minutes. It is constructed so that the lowest part of the Eye reaches ground level, enabling passengers to simply walk on to, and off of, the ride. Find a sinsuoid which models the height h of the passenger above the ground in meters t minutes after they board the Eye at ground level.

Properties of the Sinusoid $S(t) = A\sin(\omega t + \phi) + B$

- The **amplitude** is $|A|$
- The **angular frequency** is ω and the **ordinary frequency** is $f = \frac{\omega}{2\pi}$
- The **period** is $T = \frac{1}{f} = \frac{2\pi}{\omega}$
- The **phase** is ϕ and the **phase shift** is $-\frac{\phi}{\omega}$
- The **vertical shift** or **baseline** is B



$$w = \frac{2\pi}{30 \text{ min}}$$

$$\boxed{w = \frac{\pi}{15} \text{ min}}$$

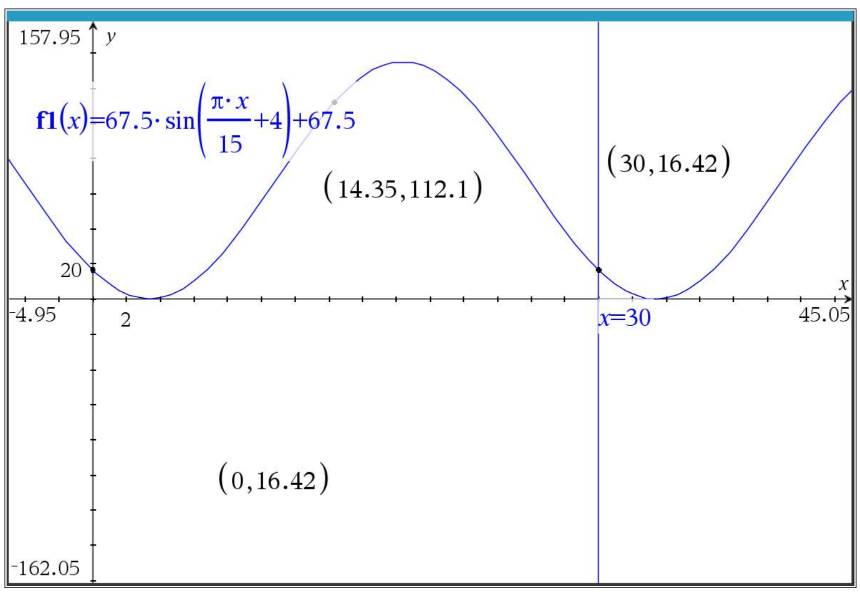
$$\boxed{B = 67.5 \text{ m}}$$

$$\boxed{A = 67.5 \text{ m}}$$

$$\boxed{\phi = 0}$$

$$h(t) = 67.5\left(\sin\left(\frac{\pi t}{15}\right)\right) + 67.5$$

$$T = \frac{2\pi}{w} = \frac{2\pi}{\frac{\pi}{15}} = 30 \text{ min}$$



This graph is not correct. Can you fix it?