12-02-25 MTH 167-002N

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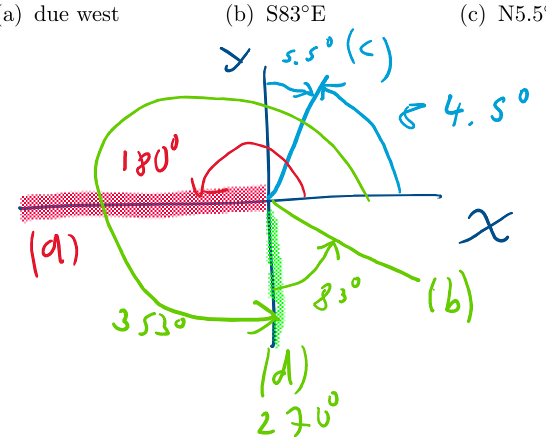
11.2
Before class notes

25. Find the angle $\theta$ in standard position with $0° \le \theta < 360°$ which corresponds to each of the bearings given below.

(a) due west   (b) S83°E   (c) N5.5°E   (d) due south



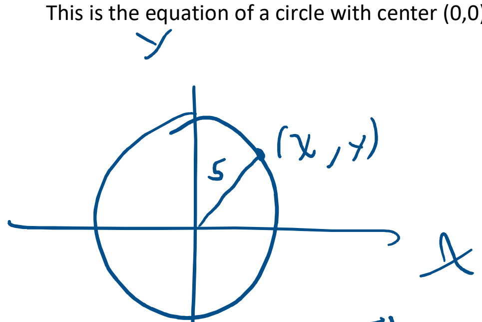

$$\begin{array}{r} 270° \\ +\ 83° \\ \hline 353° \end{array}$$

11.4: 64

In Exercises 57 - 76, convert the equation from rectangular coordinates into polar coordinates. Solve for $r$ in all but #60 through #63. In Exercises 60 - 63, you need to solve for $\theta$

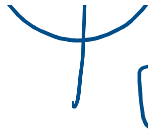
64. $x^2 + y^2 = 25$

This is the equation of a circle with center (0,0) and radius 5.




$\boxed{r = 5}$ polar equation

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = 5^2$
$\iff r^2\cos^2\theta + r^2\sin^2\theta = 5^2$
$\iff r^2(\cos^2\theta + \sin^2\theta) = 5^2$
$\iff r^2(1) = 5^2$



$\boxed{r=5}$ polar equation

$\iff r^2(\cos^2\theta+\sin^2\theta)=5^2$

$\iff r^2(1)=5^2$

$\iff r^2=5^2$

$\iff \boxed{r=5}$

74. $(x+2)^2+y^2=4$

$$(r\cos\theta+2)^2+(r\sin\theta)^2=4$$

$$r^2\cos^2\theta+4r\cos\theta+\cancel{4}+r^2\sin^2\theta=\cancel{4}$$

$$r^2\cos^2\theta+r^2\sin^2\theta+4r\cos\theta=0$$

$$r^2(\cos^2\theta+\sin^2\theta)+4r\cos\theta=0$$

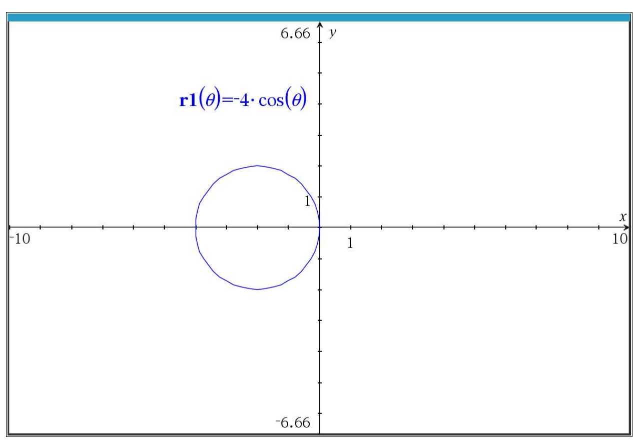
$$r^2+4r\cos\theta=0$$

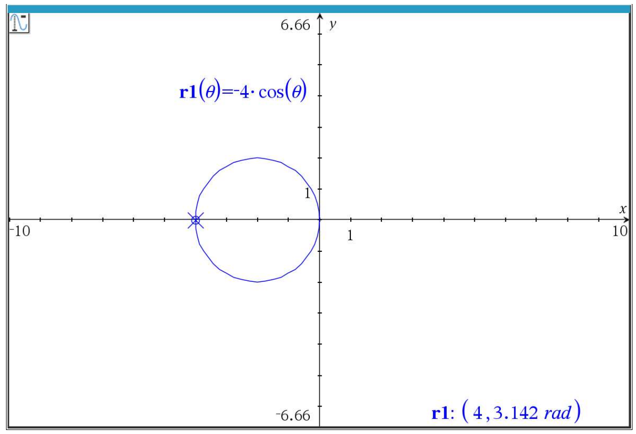
$$r(r+4\cos\theta)=0$$

$$r=0 \quad \text{or} \quad r+4\cos\theta=0$$

$$r=0 \quad \text{or} \quad \boxed{r=-4\cos\theta}$$

Note $\theta=\frac{\pi}{2} \Rightarrow r=-4\cos\left(\frac{\pi}{2}\right)=(-4)(0)=0$



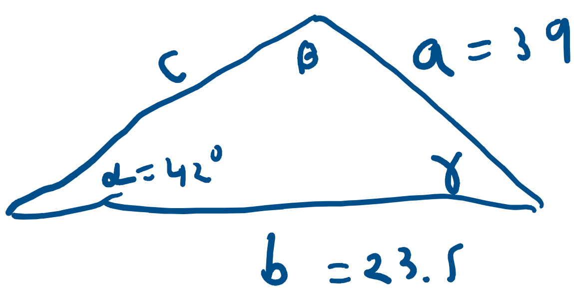



11.2

### 11.2.1 Exercises

In Exercises 1 - 20, solve for the remaining side(s) and angle(s) if possible. As in the text, $(\alpha, a)$, $(\beta, b)$ and $(\gamma, c)$ are angle-side opposite pairs.

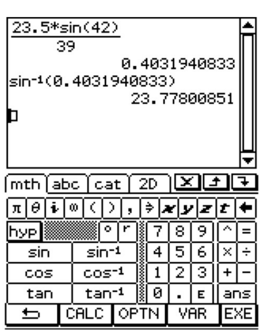
**11.** $\alpha = 42^\circ,\ a = 39,\ b = 23.5$




$$\frac{\sin \beta}{b} = \frac{\sin \alpha}{a}$$

$$\sin \beta = \frac{b \sin \alpha}{a}$$

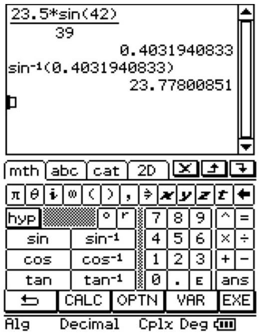
$$\sin \beta = \frac{(23.5)(\sin 42^\circ)}{39}$$




$$\boxed{\beta = 23.8^\circ}$$

180-42-23.8=114.2

$$\boxed{\gamma = 114.2^\circ}$$




$\boxed{\beta = 23.8^0}$

180-42-23.8=114.2

$\boxed{\gamma = 114.2^0}$

Textbook answer:

11. $\alpha = 42°$ $\quad \beta \approx 23.78°$ $\quad \gamma \approx 114.22°$
$a = 39$ $\quad b = 23.5$ $\quad c \approx 53.15$

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha}$$

$$c = a\,\frac{\sin\gamma}{\sin\alpha}$$

$$c = 39\,\frac{\sin 114.2^0}{\sin(42^0)}$$

39*sin(114.2)/sin(42)
53.16254285

$$\boxed{c \approx 53.2}$$

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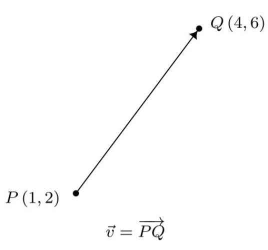
11.8

Memorize

**Definition 11.5.** Suppose $\vec{v}$ is represented by a directed line segment with initial point $P(x_0, y_0)$ and terminal point $Q(x_1, y_1)$. The **component form** of $\vec{v}$ is given by

$$\vec{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle$$

We can think of a vector as an object that has magnitude and direction.




Memorize

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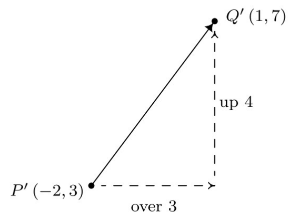


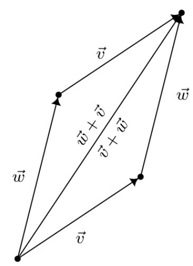

$\vec{v} = \langle 3, 4\rangle$ with initial point $P'\,(-2,3)$.

$\vec{V} = \overrightarrow{P'Q'}$
$= \langle 3, 4\rangle$

Memorize

**Definition 11.6.** Suppose $\vec{v} = \langle v_1, v_2\rangle$ and $\vec{w} = \langle w_1, w_2\rangle$. The vector $\vec{v} + \vec{w}$ is defined by

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2\rangle$$




Demonstrating the commutative property of vector addition.

Memorize

**Theorem 11.18. Properties of Vector Addition**

- **Commutative Property:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
- **Associative Property:** For all vectors $\vec{u}, \vec{v}$ and $\vec{w}$, $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
- **Identity Property:** The vector $\vec{0}$ acts as the additive identity for vector addition. That is, for all vectors $\vec{v}$,
$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}.$$
- **Inverse Property:** Every vector $\vec{v}$ has a unique additive inverse, denoted $-\vec{v}$. That is, for every vector $\vec{v}$, there is a vector $-\vec{v}$ so that
$$\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}.$$

memorize

**Definition 11.7.** If $k$ is a real number and $\vec{v} = \langle v_1, v_2\rangle$, we define $k\vec{v}$ by

$$k\vec{v} = k\langle v_1, v_2\rangle = \langle kv_1, kv_2\rangle$$

$K$ = scalar

0 1 2 ... K scale

$V + 0 = \langle V_1, V_2\rangle + \langle 0, 0\rangle$
$= \langle V_1 + 0, V_2 + 0\rangle$
$= \langle V_1, V_2\rangle = \vec{V}$

$-\vec{V} = -\langle V_1, V_2\rangle = \langle -V_1, -V_2\rangle$

Memorize

**Theorem 11.19. Properties of Scalar Multiplication**

- **Associative Property:** For every vector $\vec{v}$ and scalars $k$ and $r$, $(kr)\vec{v} = k(r\vec{v})$.
- **Identity Property:** For all vectors $\vec{v}$, $1\vec{v} = \vec{v}$.
- **Additive Inverse Property:** For all vectors $\vec{v}$, $-\vec{v} = (-1)\vec{v}$.
- **Distributive Property of Scalar Multiplication over Scalar Addition:** For every vector $\vec{v}$ and scalars $k$ and $r$,
$$(k + r)\vec{v} = k\vec{v} + r\vec{v}$$
- **Distributive Property of Scalar Multiplication over Vector Addition:** For all vectors $\vec{v}$ and $\vec{w}$ and scalars $k$,
$$k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$$
- **Zero Product Property:** If $\vec{v}$ is vector and $k$ is a scalar, then
$$k\vec{v} = \vec{0} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad \vec{v} = \vec{0}$$

$$k(\vec{v}+\vec{w}) = k\vec{v} + k\vec{w}$$

- **Zero Product Property:** If $\vec{v}$ is vector and $k$ is a scalar, then

$$k\vec{v} = \vec{0} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad \vec{v} = \vec{0}$$

Prove $k\vec{v} = \vec{0}$ iff $k=0$ or $\vec{v}=0$

Assume $k\vec{v} = \vec{0}$

Let $\vec{v} = \langle v_1, v_2 \rangle$

$k\vec{v} = \langle k v_1, k v_2 \rangle = \langle 0, 0 \rangle$

$\Rightarrow k v_1 = 0$ and $k v_2 = 0$

$(k=0 \text{ or } v_1 = 0)$ and $(k = 0 \text{ or } v_2 = 0)$

If $k=0$, we are done

If $k \neq 0$, $k v_1 = 0 \Rightarrow v_1 = 0$

$k v_2 = 0 \Rightarrow v_2 = 0$

$\therefore \vec{v} = \langle 0, 0 \rangle = \vec{0}$

The converse is easy.

Memorize

**Definition 11.8.** Suppose $\vec{v}$ is a vector with component form $\vec{v} = \langle v_1, v_2 \rangle$. Let $(r, \theta)$ be a polar representation of the point with rectangular coordinates $(v_1, v_2)$ with $r \geq 0$.

- The **magnitude** of $\vec{v}$, denoted $\|\vec{v}\|$, is given by $\|\vec{v}\| = r = \sqrt{v_1^2 + v_2^2}$
- If $\vec{v} \neq \vec{0}$, the **(vector) direction** of $\vec{v}$, denoted $\hat{v}$ is given by $\hat{v} = \langle \cos(\theta), \sin(\theta) \rangle$

Taken together, we get $\vec{v} = \langle \|\vec{v}\| \cos(\theta), \|\vec{v}\| \sin(\theta) \rangle$.

$\|\hat{v}\| = \|\langle \cos(\theta), \sin(\theta) \rangle\|$

$= \sqrt{\cos^2(\theta) + \sin^2(\theta)}$

$= \sqrt{1} = 1$

$\vec{v}$ is a unit vector, because $\|\hat{v}\| = 1$

Memorize

**Theorem 11.20. Properties of Magnitude and Direction**: Suppose $\vec{v}$ is a vector.

- $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$
- For all scalars $k$, $\|k\,\vec{v}\| = |k| \|\vec{v}\|$.
- If $\vec{v} \neq \vec{0}$ then $\vec{v} = \|\vec{v}\| \hat{v}$, so that $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right) \vec{v}$.

Let $\vec{v} = \langle 1, 2 \rangle$

Find $\hat{v}$

$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, 2 \rangle}{\sqrt{1^2 + 2^2}}$

$= \frac{\langle 1, 2 \rangle}{\sqrt{5}}$

$\boxed{\hat{v} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle}$

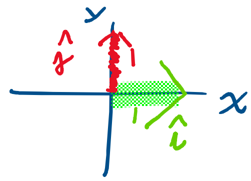
Memorize

**Definition 11.9. Unit Vectors:** Let $\vec{v}$ be a vector. If $\|\vec{v}\| = 1$, we say that $\vec{v}$ is a **unit vector**.

Memorize

**Definition 11.10. The Principal Unit Vectors:**

- The vector $\hat{\imath}$ is defined by $\hat{\imath} = \langle 1, 0\rangle$
- The vector $\hat{\jmath}$ is defined by $\hat{\imath}$ $\hat{\jmath}$ $= \langle 0, 1\rangle$




Memorize

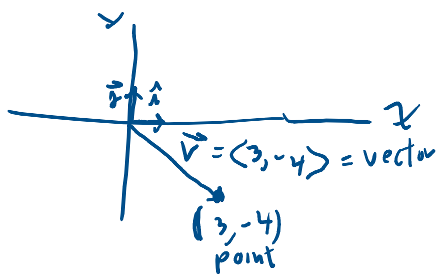
**Theorem 11.21. Principal Vector Decomposition Theorem:** Let $\vec{v}$ be a vector with component form $\vec{v} = \langle v_1, v_2\rangle$. Then $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$.

Let $\vec{v} = \langle 3, -4\rangle$

$= \langle 3, 0\rangle + \langle 0, -4\rangle$

$= 3\langle 1, 0\rangle - 4\langle 0, 1\rangle$

$= 3\hat{\imath} - 4\hat{\jmath}$




11.9

Memorize

**Definition 11.11.** Suppose $\vec{v}$ and $\vec{w}$ are vectors whose component forms are $\vec{v} = \langle v_1, v_2\rangle$ and $\vec{w} = \langle w_1, w_2\rangle$. The **dot product** of $\vec{v}$ and $\vec{w}$ is given by

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2\rangle \cdot \langle w_1, w_2\rangle = v_1w_1 + v_2w_2$$

Let $\vec{v} = \langle 5, 2\rangle$

$\vec{w} = \langle 6, 3\rangle$

$\vec{v} \cdot \vec{w} = (5)(6) + (2)(3) = 30 + 6 = \boxed{36}$

Memorize

**Theorem 11.22. Properties of the Dot Product**

- **Commutative Property:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
- **Distributive Property:** For all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
- **Scalar Property:** For all vectors $\vec{v}$ and $\vec{w}$ and scalars $k$, $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (k\vec{w})$.
- **Relation to Magnitude:** For all vectors $\vec{v}$, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

Proof of commutative property

Let $\vec{v} = \langle v_1, v_2\rangle$

$\vec{w} = \langle w_1, w_2\rangle$

$\vec{v} \cdot \vec{w} = \langle v_1, v_2\rangle \cdot \langle w_1, w_2\rangle$

$= v_1w_1 + v_2w_2$ (by def of dot product)

$= w_1v_1 + w_2v_2$ (by commutative property of mult. of real numbers)

$= v_1w_1 + v_2w_2$ (by def of dot product)

$= w_1v_1 + w_2v_2$ (by cummutative property of mult. of real numbers)

$= \vec{w} \cdot \vec{v}$ (by def of dot product

9

Prove $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

let $\vec{v} = \langle v_1, v_2 \rangle$

$\vec{v} \cdot \vec{v} = \langle v_1, v_2 \rangle \cdot \langle v_1, v_2 \rangle$

$= v_1^2 + v_2^2$

$= \|\vec{v}\|^2$

Memorize

**Theorem 11.23. Geometric Interpretation of Dot Product:** If $\vec{v}$ and $\vec{w}$ are nonzero vectors then $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$, where $\theta$ is the angle between $\vec{v}$ and $\vec{w}$.

let $\theta = \frac{\pi}{2}$


$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\left(\frac{\pi}{2}\right)$

$= \|\vec{v}\| \|\vec{w}\| (0)$

$= 0$

Memorize

**Theorem 11.24.** Let $\vec{v}$ and $\vec{w}$ be nonzero vectors and let $\theta$ the angle between $\vec{v}$ and $\vec{w}$. Then

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos(\hat{v} \cdot \hat{w})$$

Memorize

**Theorem 11.25. The Dot Product Detects Orthogonality:** Let $\vec{v}$ and $\vec{w}$ be nonzero vectors. Then $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$.

Supplied

**Definition 11.12.** Let $\vec{v}$ and $\vec{w}$ be nonzero vectors. The **orthogonal projection of** $\vec{v}$ **onto** $\vec{w}$, denoted $\text{proj}_{\vec{w}}(\vec{v})$ is given by $\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \hat{w})\hat{w}$.

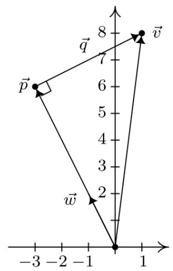
Supplied

**Theorem 11.26. Alternate Formulas for Vector Projections:** If $\vec{v}$ and $\vec{w}$ are nonzero vectors then

$$\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \hat{w})\hat{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\right)\vec{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}$$

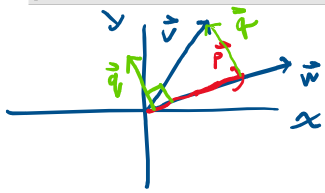
**Example 11.9.4.** Let $\vec{v} = \langle 1, 8 \rangle$ and $\vec{w} = \langle -1, 2 \rangle$. Find $\vec{p} = \text{proj}_{\vec{w}}(\vec{v})$, and plot $\vec{v}$, $\vec{w}$ and $\vec{p}$ in standard position.

**Solution.** We find $\vec{v} \cdot \vec{w} = \langle 1, 8 \rangle \cdot \langle -1, 2 \rangle = (-1) + 16 = 15$ and $\vec{w} \cdot \vec{w} = \langle -1, 2 \rangle \cdot \langle -1, 2 \rangle = 1 + 4 = 5$. Hence, $\vec{p} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w} = \frac{15}{5}\langle -1, 2 \rangle = \langle -3, 6 \rangle$. We plot $\vec{v}$, $\vec{w}$ and $\vec{p}$ below.




supplied

**Theorem 11.27. Generalized Decomposition Theorem:** Let $\vec{v}$ and $\vec{w}$ be nonzero vectors. There are unique vectors $\vec{p}$ and $\vec{q}$ such that $\vec{v} = \vec{p} + \vec{q}$ where $\vec{p} = k\vec{w}$ for some scalar $k$, and $\vec{q} \cdot \vec{w} = 0$.




Supplied

**Theorem 11.28. Work as a Dot Product:** Suppose a constant force $\vec{F}$ is applied along the vector $\overrightarrow{PQ}$. The work $W$ done by $\vec{F}$ is given by

$$W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\| \|\overrightarrow{PQ}\| \cos(\theta),$$

where $\theta$ is the angle between $\vec{F}$ and $\overrightarrow{PQ}$.

8.1 Solve the system of linear equations

$$2x + y = 3$$
$$x - y = 1$$

Substitution, elimination

$$x = y + 1$$

substitute

$$2(y+1) + y = 3$$
$$2y + 2 + y = 3$$
$$3y = 1$$
$$\boxed{y = \frac{1}{3}}$$
$$x = \frac{1}{3} + 1 = \boxed{\frac{4}{3} = x}$$

$$\left.\begin{aligned} 2x + y &= 3 \\ x - y &= 1 \end{aligned}\right\}$$ Add the equations to eliminate y

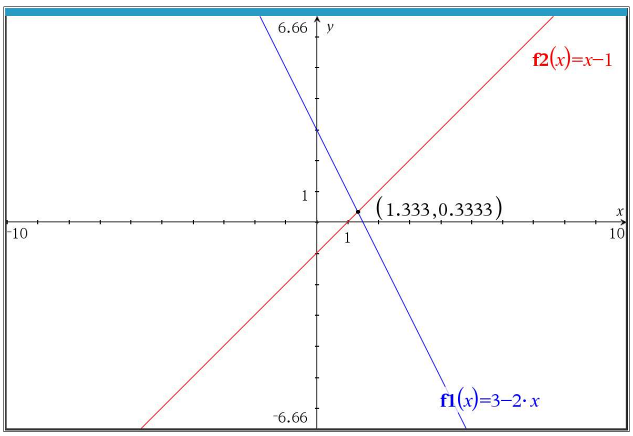
$$3x = 4$$
$$\boxed{x = \frac{4}{3}}$$

$$\left.\begin{aligned} 2x + y &= 3 \\ 2x - 2y &= 2 \end{aligned}\right\}$$ subtract 2nd from 1st to eliminate $x$

$$3y = 1$$

$$\boxed{y = \frac{1}{3}}$$

$$2x + y = 3 \Rightarrow y = 3 - 2x$$
$$x - y = 1 \Rightarrow y = x - 1$$




8.1
Memorize

**Theorem 8.1.** Given a system of equations, the following moves will result in an equivalent system of equations.

- Interchange the position of any two equations.
- Replace an equation with a nonzero multiple of itself.[a]
- Replace an equation with itself plus a nonzero multiple of another equation.

[a]That is, an equation which results from multiplying both sides of the equation by the same nonzero number.

8.2

$$2x + y = 3$$
$$x - y = 1$$

Augmented matrix

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \end{array}\right] R1 - 2R2$$

Goal

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \quad R1-2R2 \qquad \left[\begin{array}{cc|c} 0 & 1 & b \end{array}\right]$$

$$1\cdot x + 0\cdot y = a$$
$$0\cdot x + 1\cdot y = b$$
$$x = a$$
$$y = b$$

$$\left[\begin{array}{cc|c} 2-2(1) & 1-2(-1) & 3-2(1) \\ 1 & -1 & 1 \end{array}\right]$$

$$\left(\begin{array}{cc|c} 0 & 3 & 1 \\ 1 & -1 & 1 \end{array}\right) \quad \text{Interchange } R_1, R_2$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 1 \end{array}\right] \quad \frac{R2}{3}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & \frac{1}{3} \end{array}\right) \quad R1+R2$$

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} \end{array}\right) \quad \text{rref}$$

reduced row echelon form

$$\boxed{\begin{array}{l} x = \frac{4}{3} \\ y = \frac{1}{3} \end{array}}$$

Showed how to compute rref on TI.