10-09-25 MTH 167-002N

4 Rational Functions

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4.1:19

19. The cost C in dollars to remove $p\%$ of the invasive species of Ippizuti fish from Sasquatch Pond is given by

$$C(p) = \frac{1770p}{100-p}, \quad 0 \le p < 100$$

(a) Find and interpret $C(25)$ and $C(95)$.

P

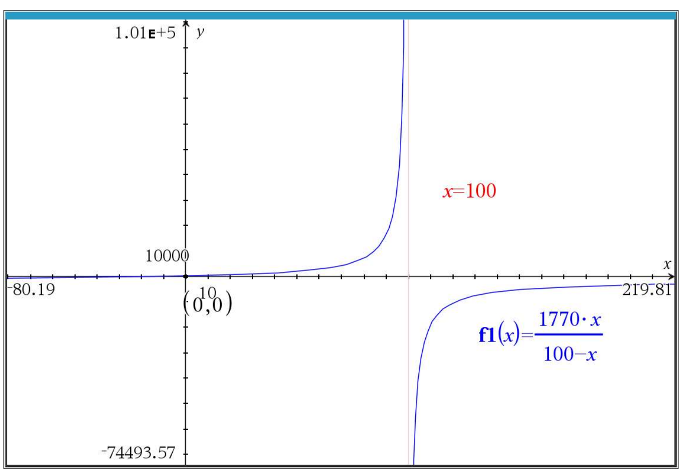
(b) What does the vertical asymptote at $x = 100$ mean within the context of the problem?

(c) What percentage of the Ippizuti fish can you remove for \$40000?

(a) $c(25) = \dfrac{1770(25)}{100-25} = \boxed{590}$

1770*25/(100-25)=590
1770*25/100-25=417.5 (wrong, use PEMDAS)

interpret
It costs \$590 to remove 25% of the fish.



$c(p) = 40000$
Find p

$$\frac{1770p}{100-p} = 40{,}000$$

$$\frac{1770p}{100-p} = 40{,}000$$

Gemini

2.5 Flash

The cost C in dollars to remove $p\%$ of the invasive species of Ippizuti fish from Sasquatch Pond is given by

$$C(p) = \frac{1770p}{100-p}, \quad 0 \leq p < 100$$

(a) Find and interpret $C(25)$ and $C(95)$.
(b) What does the vertical asymptote at $\overset{p}{x} = 100$ mean within the context of the problem?
(c) What percentage of the Ippizuti fish can you remove for \$40000?

Analysis

This is a problem involving a rational function that models the cost of removing an invasive fish species. I'll break down the solution into the three parts requested.

The cost C in dollars to remove $p\%$ of the invasive species of Ippizuti fish from Sasquatch Pond is given by:

$$C(p) = \frac{1770p}{100-p}, \quad 0 \leq p < 100$$

(a) Find and Interpret $C(25)$ and $C(95)$

To find $C(25)$ and $C(95)$, we substitute $p = 25$ and $p = 95$ into the function.

Calculation of $C(25)$

$$C(25) = \frac{1770(25)}{100-25} = \frac{44250}{75} = 590$$

Interpretation of $C(25)$:
$C(25) = 590$ means that the **cost to remove** 25% of the Ippizuti fish from Sasquatch Pond is $\$590$.

Calculation of $C(95)$

$$C(95) = \frac{1770(95)}{100-95} = \frac{168150}{5} = 33630$$

Interpretation of $C(95)$:
$C(95) = 33630$ means that the **cost to remove** 95% of the Ippizuti fish from Sasquatch Pond is $\$33,630$.

(b) What does the vertical asymptote at $p = 100$ mean within the context of the problem?

A **vertical asymptote** occurs where the denominator of a rational function is zero, provided the numerator is not also zero at that value. In this case, the denominator is $100 - p$.
Setting the denominator to zero:

$$100 - p = 0 \implies p = 100$$

Since the numerator, $1770p$, is $1770(100) = 177,000 \neq 0$ when $p = 100$, the function has a vertical asymptote at $p = 100$.

The vertical asymptote at $p = 100$ means that as the percentage of fish removed, p, gets **closer and closer to** 100%, the cost $C(p)$ **increases without bound** (approaches infinity).

Interpretation in Context:
It implies that it is **impossible** to remove *exactly* 100% of the invasive Ippizuti fish, or that doing so would require an **infinite** (or prohibitively large) cost. The closer you get to complete removal, the exponentially more expensive the operation becomes.

(c) What percentage of the Ippizuti fish can you remove for $\$40,000$?

Here, we are given the cost $C(p) = 40000$ and need to solve for the percentage p.

$$\$40,000 = \frac{1770p}{100 - p}$$

1. **Multiply both sides** by the denominator $(100 - p)$:

$$40000(100 - p) = 1770p$$

2. **Distribute** on the left side:

$$4000000 - 40000p = 1770p$$

3. **Add** $40000p$ to both sides to gather the p terms:

$$4000000 = 1770p + 40000p$$

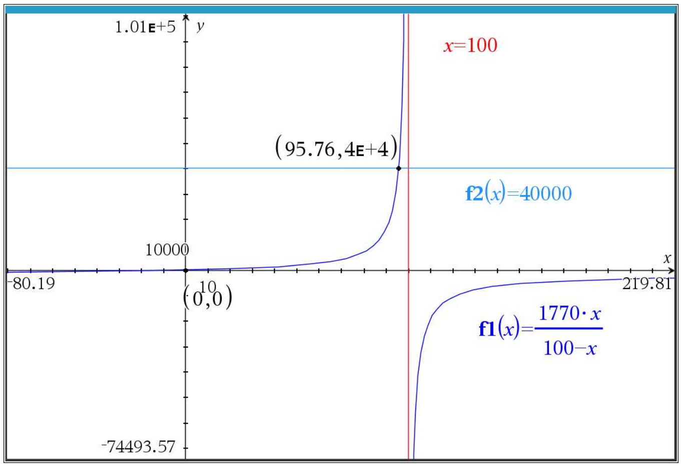
$$4000000 = 41770p$$

4. **Solve for** p:

$$p = \frac{4000000}{41770} \approx 95.7625\%$$

Rounding to two decimal places, we get $p \approx 95.76\%$.

For $\$40,000$, you can remove approximately 95.76% of the Ippizuti fish. This makes sense as it's greater than the 95% removal cost of $\$33,630$ we calculated in part (a).



$p \approx 95.76$

For $40,000, we can remove about 95.76% of the fish.

4.2
Memorize this technique

Steps for Constructing a Sign Diagram for a Rational Function

Suppose r is a rational function.

1. Place any values excluded from the domain of r on the number line with an '‽' above them.
2. Find the zeros of r and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine the sign of $r(x)$ for each test value in step 3, and write that sign above the corresponding interval.

Steps for Graphing Rational Functions

Suppose r is a rational function.

1. Find the domain of r.
2. Reduce $r(x)$ to lowest terms, if applicable. or check for holes/vertical asymptotes
3. Find the x- and y-intercepts of the graph of $y = r(x)$, if they exist.
4. Determine the location of any vertical asymptotes or holes in the graph, if they exist. Analyze the behavior of r on either side of the vertical asymptotes, if applicable.
5. Analyze the end behavior of r. Find the horizontal or slant asymptote, if one exists.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph of $y = r(x)$.

4.2

4.2.1 Exercises

In Exercises 1 - 16, use the six-step procedure to graph the rational function. Be sure to draw any asymptotes as dashed lines.

4. $f(x) = \dfrac{1}{x^2 + x - 12}$

$x^2 + x - 12 = (x+4)(x-3)$

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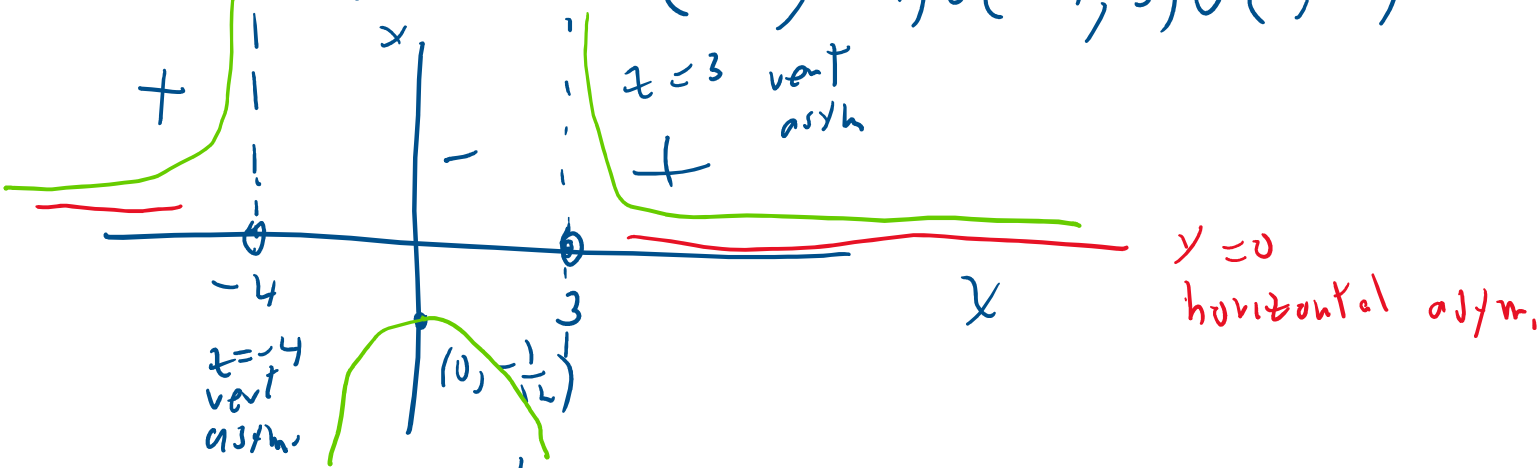
$$x^2 + x - 12 = (x+4)(x-3)$$

$\Rightarrow$ domain $= (-\infty, -4) \cup (-4, 3) \cup (3, \infty)$



x-intercept

set $y = 0$ solve for x

$$\frac{1}{x^2 + x - 12} = 0$$

$$1 \neq 0$$

$\therefore$ no x-intercept

y-intercept $f(0) = \frac{1}{0^2 + 0 - 12} = -\frac{1}{12}$

$$f(0) = -\frac{1}{12} < 0$$

$$f(-5) = \frac{1}{(-5)^2 - 5 - 12} = \frac{1}{25 - 17} > 0$$

$$f(4) = \frac{1}{4^2 + 4 - 12} = \frac{1}{20 - 12} > 0$$

horizontal or slant asymptote?

no x^3 in numerator $\Rightarrow$ no slant asym.

deg 1 < 2 $\Rightarrow$ horizontal asym $y = 0$

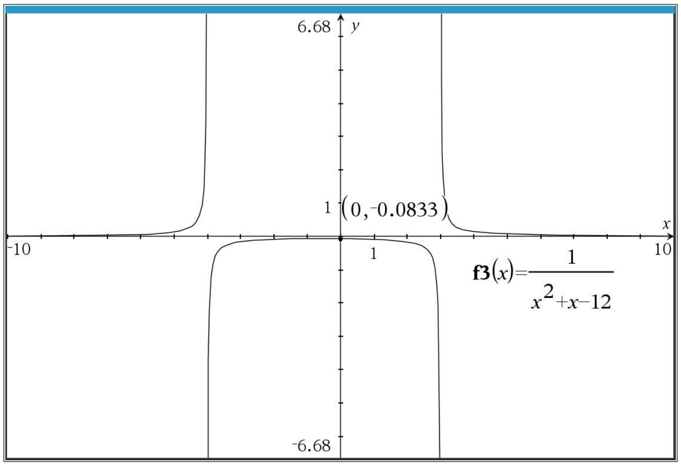
no x in num...

deg 1 < 2 ⇒ horizontal asym $y = 0$

end behavior

$x \to -\infty \Rightarrow f(x) \to 0$

$x \to \infty \Rightarrow f(x) \to 0$



4.3

Memorize

Definition 4.5. Suppose x, y and z are variable quantities. We say

- y **varies directly with** (or is **directly proportional to**) x if there is a constant k such that $y = kx$.
- y **varies inversely with** (or is **inversely proportional to**) x if there is a constant k such that $y = \frac{k}{x}$. $\Leftrightarrow$ $xy = k$
- z **varies jointly with** (or is **jointly proportional to**) x and y if there is a constant k such that $z = kxy$.

The constant k in the above definitions is called the **constant of proportionality**.

Example 4.3.6. Translate the following into mathematical equations using Definition 4.5.

1. Hooke's Law: The force F exerted on a spring is directly proportional the extension x of the spring.

$F = kx$

4.3: 3

In Exercises 1 - 6, solve the rational equation. Be sure to check for extraneous solutions.

3. $\dfrac{1}{x+3} + \dfrac{1}{x-3} = \dfrac{x^2-3}{x^2-9} = \dfrac{x^2-3}{(x+3)(x-3)}$

Note! $x \neq \pm 3$

$x+3$ $x-3$ x^2-9

$(x+3)(x-3)$

$$\left(\frac{1}{\cancel{x+3}}\right)(\cancel{x+3})(x-3) + \left(\frac{1}{\cancel{x-3}}\right)(x+3)(\cancel{x-3}) = \frac{(x^2-3)}{(\cancel{x+3})(\cancel{x-3})}(\cancel{x+3})(\cancel{x-3})$$

$$x-3 + x+3 = x^2-3$$

$$2x = x^2-3$$

$$x^2-2x-3 = 0$$

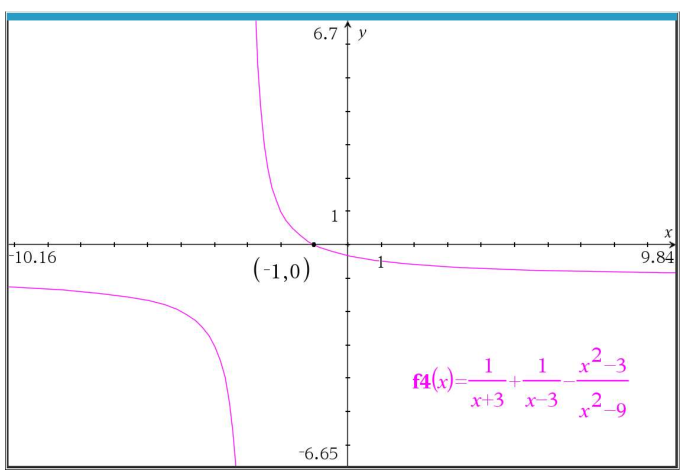
$$(x-3)(x+1) = 0$$

$$\boxed{x = 3, -1}$$

$x \neq \pm 3 \Rightarrow x = 3$ is an extraneous solution

$$\therefore \boxed{x = -1}$$

The graph agrees with our manual (analytic) calculation.



In Exercises 7 - 20, solve the rational inequality. Express your answer using interval notation.

15. $\dfrac{3x-1}{x^2+1} \leq 1$

$$\frac{3x-1}{x^2+1} - 1 \leq 0$$

$$x^2+1 \neq 0$$

solve $\frac{3x-1}{x^2+1} - 1 = 0$

$$\Rightarrow \left(\frac{3x-1}{x^2+1}\right)(x^2+1) - 1(x^2+1) = 0(x^2+1)$$

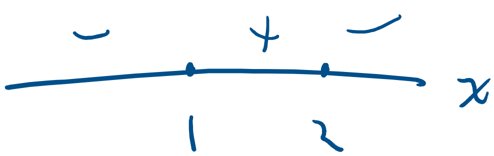
$$3x-1-x^2-1=0$$

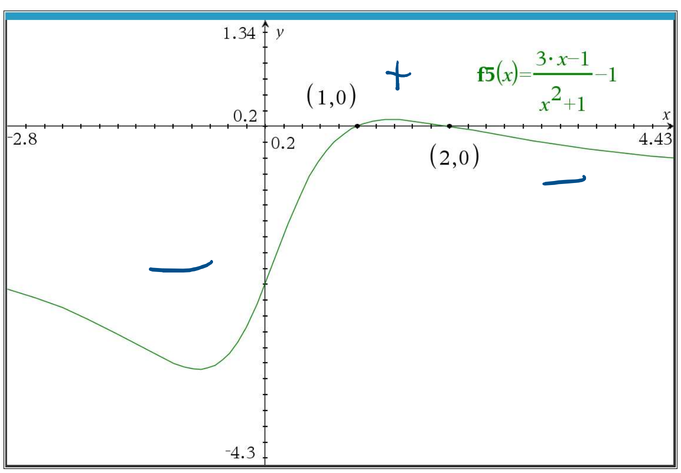
$$-x^2+3x-2=0$$

$$x^2-3x+2=0$$

$$(x-2)(x-1)=0$$

$$\boxed{x=2,1}$$





solution set

$[1, 2]$

14. $\frac{x^2+5x+6}{x^2-1} > 0$

Let $f(x) = \frac{x^2+5x+6}{x^2-1}$

solve $f(x)>0$

$$f(x) = \frac{(x+3)(x+2)}{(x+1)(x-1)}$$

$x \neq \pm 1$

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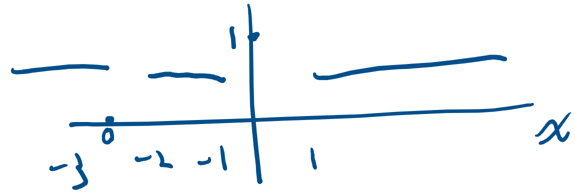
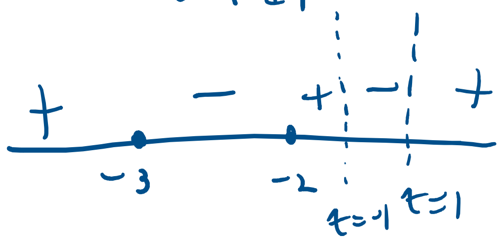
$y_1 = \frac{x^2+5x+6}{x^2-1} > 0$



$(x+1)(x-1)$

$x \neq \pm 1$



solution set

$(-\infty, -3) \cup (-2, -1) \cup (1, \infty)$

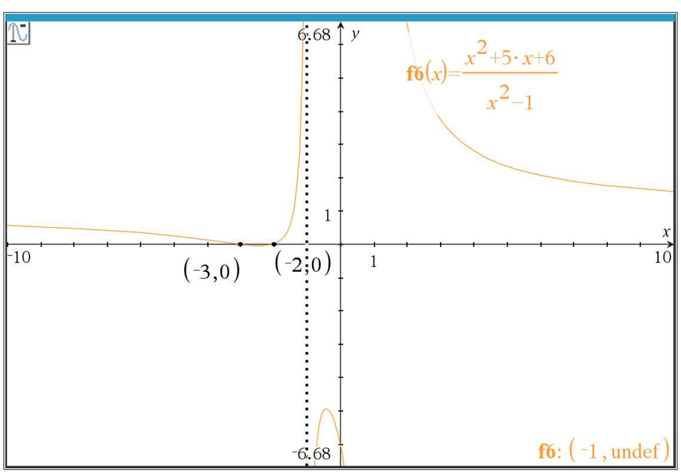
$$f(-4) = \frac{(-4+3)(-4+2)}{(-4+1)(-4-1)} = \frac{(-)(-)}{(-)(-)} > 0$$

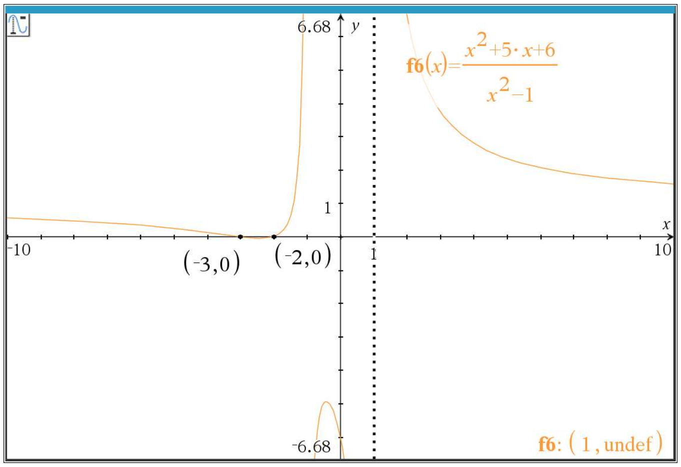
$$f\left(-\frac{5}{2}\right) = \frac{\left(-\frac{5}{2}+3\right)\left(-\frac{5}{2}+2\right)}{\left(-\frac{5}{2}+1\right)\left(-\frac{5}{2}-1\right)} = \frac{(+)(-)}{(-)(-)} < 0$$

$$f\left(-\frac{3}{2}\right) = \frac{\left(-\frac{3}{2}+3\right)\left(-\frac{3}{2}+2\right)}{\left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}-1\right)} = \frac{(+)(+)}{(-)(-)} > 0$$

$$f(0) = \frac{(0+3)(0+2)}{(0+1)(0-1)} = \frac{(+)(+)}{(+)(-)} < 0$$

$$f(2) = \frac{(2+3)(2+2)}{(2+1)(2-1)} = \frac{(+)(+)}{(+)(+)} > 0$$





6.68
y
$f6(x)=\frac{x^2+5\cdot x+6}{x^2-1}$
1
x
-10
1
10
(-3,0)
(-2,0)
-6.68
f6: (1, undef)