10-02-25 MTH 167-002N

3.2 The Factor Theorem and the Remainder Theorem

3.2.1 Exercises

page 265 (277): 1, 3, 9, 21, 35, 42

3.3 Real Zeros of Polynomials

3.3.3: Exercises

page 280 (392): 1, 31, 37, 48

3.4 Complex Zeros and the Fundamental Theorem of Algebra

3.4.1 Exercises

page 295 (307): 1, 11, 13, 23, 27, 50

3.2: 35

In Exercises 31 - 40, you are given a polynomial and one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

35. $x^3 + 2x^2 - 3x - 6, \;\; c = -2$

Although there is a formula for solving cubic polynomial equations, we will not use it in this class.

$c = -2$ is a zero $\Rightarrow$ $x - (-2) = x + 2$ is a factor of the cubic function

$$(x^3 + 2x^2 - 3x - 6) \div (x + 2)$$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -3 & -6 \\ & & -2 & 0 & 6 \\ \hline & 1 & 0 & -3 & (0) \end{array}$$

$x^2 - 3$

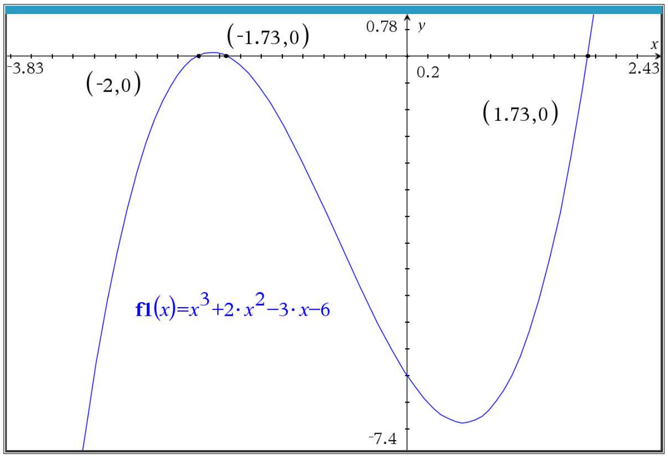
$$x^3 + 2x^2 - 3x - 6 = (x+2)(x^2 - 3)$$

$$x^2 - 3 = 0$$

$$x^2 = 3$$

$$\boxed{x = \pm\sqrt{3}}$$

Sqrt(3)=1.732050807568877



3.2:42

In Exercises 41 - 45, create a polynomial p which has the desired characteristics. You may leave the polynomial in factored form.

42.
- The zeros of p are $c = 1$ and $c = 3$
- $c = 3$ is a zero of multiplicity 2.
- The leading term of $p(x)$ is $-5x^3$

$$p(x) = a(x-1)^m (x-3)^n \quad n=2$$

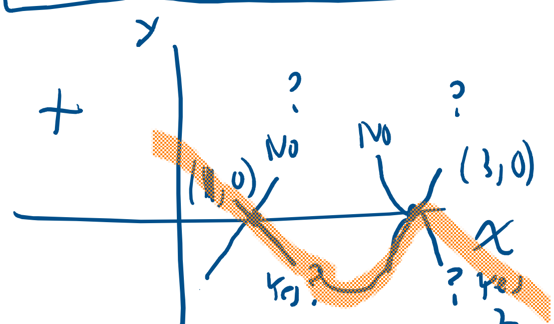
a = constant $= -5$

$$p(x) = -5(x-1)^m (x-3)^2$$

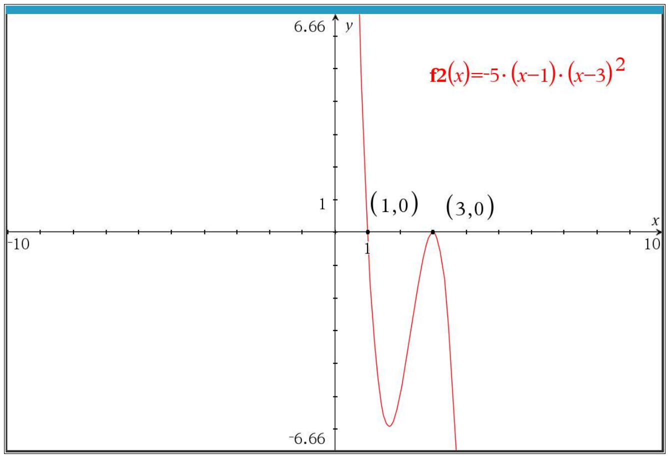
leading term is $-5x^3$

$\Rightarrow \deg p(x) = 3 \Rightarrow m = 1$

$$\boxed{p(x) = -5(x-1)(x-3)^2}$$

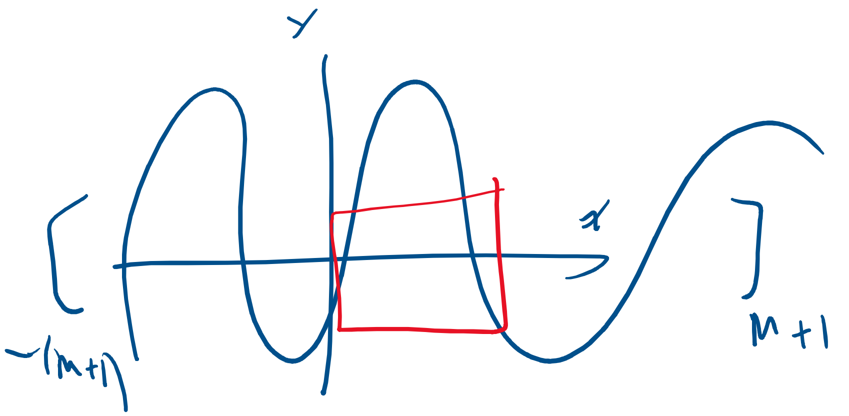


$$p(0) = -5(0-1)(0-3)^2$$
$$= -5(-1)(9)$$
$$= 45 > 0$$



3.3
supplied

Theorem 3.8. Cauchy's Bound: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}$. Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.



Let $f(x) = 4x^5 + 6x^4 - 2x^2 + 1$

$a_n = a_5 = 4$

$a_4 = 6$

$a_3 = 0$

$a_2 = -2$

$a_1 = 0$

$a_0 = 1$

$M = \max\left\{ \frac{|1|}{|4|}, \frac{|0|}{|4|}, \frac{|-2|}{|4|}, \frac{|0|}{|4|}, \frac{|6|}{|4|} \right\}$

$= \max\left\{ \frac{1}{4}, 0, \frac{1}{2}, 0, \frac{3}{2} \right\}$

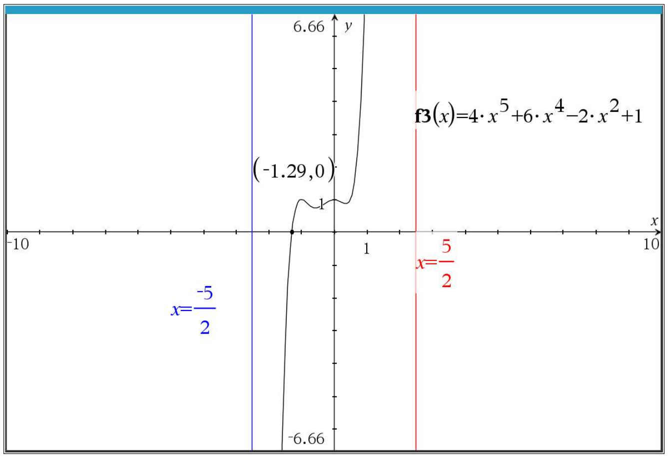
$M = \frac{3}{2}$

$M+1 = 5$

$m = \frac{3}{2}$

$\boxed{m+1 = \frac{5}{2}}$

All real zeros are in $\left(-\frac{5}{2}, \frac{5}{2}\right)$



solve$(4\cdot x^5+6\cdot x^4-2\cdot x^2+1=0,x)$ $x=-1.28977$

cSolve$(4\cdot x^5+6\cdot x^4-2\cdot x^2+1=0,x)$

...125−0.314082·i or $x=$−0.598239+0.457353·i or $x=$−0.598239−0.457353·i or $x=$−1.28977

$a_1 + b_1 i \qquad a_1 - b_1 i$

cSolve$(4\cdot x^5+6\cdot x^4-2\cdot x^2+1=0,x)$

$x=$0.493125+0.314082·i or $x=$0.493125−0.314082·i or $x=$−0.598239+0.457353·i or $x=$−0...

$a_2 + b_2 i \qquad a_2 - b_2 i$

Supplied

Theorem 3.9. Rational Zeros Theorem: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and $a_0, a_1, \ldots a_n$ are integers. If r is a rational zero of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

$f(x) = (3x-1)(x+3)(x^2-2)$

zeros: $\underbrace{\frac{1}{3}, -3,}_{\text{rational}} \underbrace{\pm\sqrt{2}}_{\text{irrational}}$

$f(x) = (3x^2+8x-3)(x^2-2)$

$= 3x^4 - 6x^2 + 8x^3 - 16x - 3x^2 + 6$

$f(x) = 3x^4 + 8x^3 - 9x^2 - 16x + 6$

$$\boxed{f(x) = 3x^4 + 8x^3 - 9x^2 - 16x + 6}$$

$a | b$ means a divides b
or a is a factor of b

rational root has the form $\frac{p}{q}$

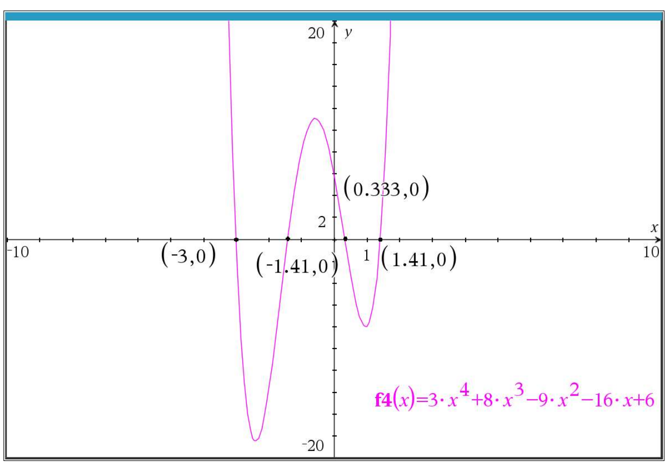
$p | a_0 \quad q | a_n$

$p | 6 \quad q | 3$

$p = \pm 1, \pm 2, \pm 3, \pm 6$

$q = \pm 1, \pm 3$

$$\frac{p}{q} = \pm 1, \underset{.333}{\pm \frac{1}{3}}, \pm 2, \pm \frac{2}{3}, \underset{-3}{\pm 3}, \pm 6$$



From graph
$x \approx -3, \pm 1.41, .333$

check $\left.\begin{array}{l} f\left(\frac{1}{3}\right) = 0 \\ f(-3) = 0 \end{array}\right\}$ 2 rational zeros

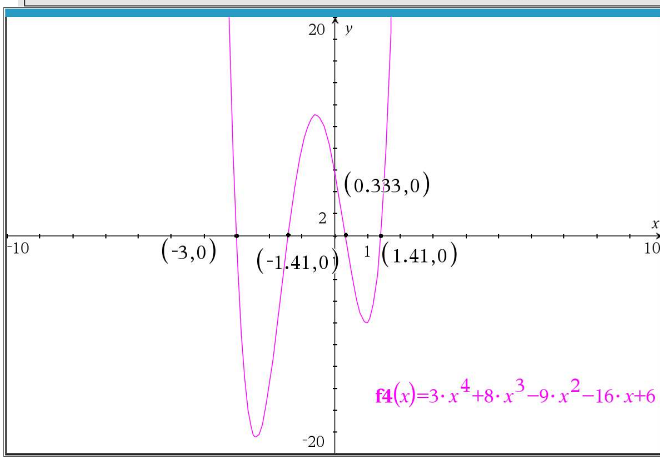
$x = \pm 1.41$ are <u>not</u> in our list
of possible rational zeros.

$\therefore x \approx 1.41$ are irrational

Supplied

Theorem 3.10. Descartes' Rule of Signs: Suppose $f(x)$ is the formula for a polynomial function written with descending powers of x.

- If P denotes the number of variations of sign in the formula for $f(x)$, then the number of positive real zeros (counting multiplicity) is one of the numbers $\{P,\ P-2,\ P-4,\ \ldots\}$.
- If N denotes the number of variations of sign in the formula for $f(-x)$, then the number of negative real zeros (counting multiplicity) is one of the numbers $\{N,\ N-2,\ N-4,\ \ldots\}$.



$$f(x) = 3x^4 \underbrace{+\,8x^3}_{0}\ \underbrace{-\,9x^2}_{1}\ \underbrace{-\,16x}_{0}\ \underbrace{+\,6}_{1}$$

2 sign changes $\Rightarrow$ 2 or 0 positive real zeros

$$f(-x) = 3(-x)^4 + 8(-x)^3 - 9(-x)^2 - 16(-x) + 6$$

$$f(-x) = 3x^4 \underbrace{-\,8x^3}_{1}\ \underbrace{-\,9x^2}_{0}\ \underbrace{+\,16x}_{1}\ \underbrace{+\,6}_{0}$$

2 sign changes $\Rightarrow$ 2 or 0 negative real zeros

supplied

Theorem 3.11. Upper and Lower Bounds: Suppose f is a polynomial of degree $n \geq 1$.

- If $c > 0$ is synthetically divided into f and all of the numbers in the final line of the division tableau have the same signs, then c is an upper bound for the real zeros of f. That is, there are no real zeros greater than c.
- If $c < 0$ is synthetically divided into f and the numbers in the final line of the division tableau alternate signs, then c is a lower bound for the real zeros of f. That is, there are no real zeros less than c.

  NOTE: If the number 0 occurs in the final line of the division tableau in either of the above cases, it can be treated as $(+)$ or $(-)$ as needed.

3.4

Memorize

Definition 3.4. The imaginary unit i satisfies the two following properties

1. $i^2 = -1$ $\Rightarrow i = \sqrt{-1}$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

$$\sqrt{-c} = \sqrt{-1}\sqrt{c}$$
$$= i\sqrt{c}$$

Memorize

Definition 3.5. A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit.

$\mathbb{C}$ = set of complex numbers

$$\mathbb{C} = \{a + bi \mid a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1\}$$

$$a \in \mathbb{R} \Rightarrow a = a + 0i \in \mathbb{C}$$

complex arithmetic

$$(2+3i) + (6-i) = 8 + 2i$$
$$= (2+6) + (3i - i)$$
$$= 8 + i(3-1)$$
$$= 8 + 2i$$

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

$$(2+i)(6-i)$$
$$= (2)(6) + (2)(-i) + 6i - i^2$$
$$= 12 + 4i - (-1)$$
$$= 12 + 1 + 4i$$
$$= \boxed{13 + 4i}$$

$$\frac{13+4i}{2+i} \cdot \left(\frac{2-i}{2-i}\right)$$

$$\boxed{(a+b)(a-b) = a^2 - b^2}$$

$$\frac{13+4i}{2+i}\cdot\left(\frac{2-i}{2-i}\right)$$

$$= \frac{26-5i-4i^2}{4-(i^2)}$$

$$= \frac{26-5i-4(-1)}{4-(-1)}$$

$$= \frac{30-5i}{5} = 6-i$$

$= a^2 - b^2$

$\frac{13+4\cdot \boldsymbol{i}}{2+\boldsymbol{i}}$	$6-\boldsymbol{i}$

Memorize Definition:

Let $z = a+bi \in \mathbb{C}$ = set of complex numbers

then $\bar{z} = a-bi$ = conjugate of z

$x \in A$ means x is an element of set A

Supplied or I will ask you to prove

Theorem 3.12. Properties of the Complex Conjugate: Let z and w be complex numbers.

- $\overline{\overline{z}} = z$
- $\overline{z} + \overline{w} = \overline{z+w}$
- $\overline{z}\,\overline{w} = \overline{zw}$
- $(\overline{z})^n = \overline{z^n}$, for any natural number n
- z is a real number if and only if $\overline{z} = z$.

Supplied

Theorem 3.13. The Fundamental Theorem of Algebra: Suppose f is a polynomial function with complex number coefficients of degree $n \geq 1$, then f has at least one complex zero.

Supplied

Theorem 3.14. Complex Factorization Theorem: Suppose f is a polynomial function with complex number coefficients. If the degree of f is n and $n \geq 1$, then f has exactly n complex zeros, counting multiplicity. If $z_1, z_2, \ldots, z_k$ are the distinct zeros of f, with multiplicities $m_1, m_2, \ldots, m_k$, respectively, then $f(x) = a\,(x-z_1)^{m_1}\,(x-z_2)^{m_2}\cdots(x-z_k)^{m_k}$.

Memorize

Theorem 3.15. Conjugate Pairs Theorem: If f is a polynomial function with real number coefficients and z is a zero of f, then so is $\overline{z}$.

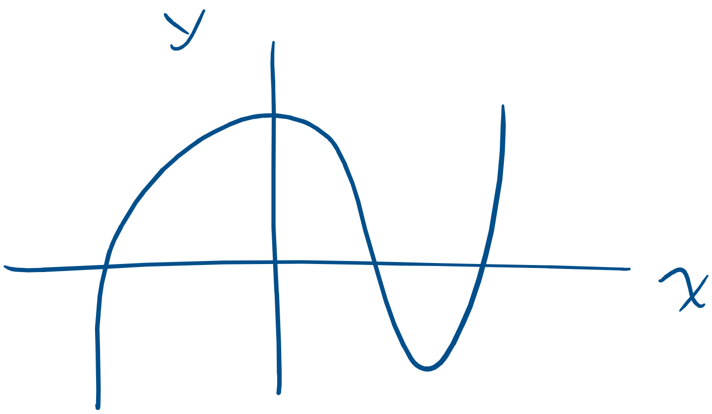
Let $z = 3 = 3 + 0i \in \mathbb{C}$

Let $z = 3 = 3 + 0i \in \mathbb{C}$

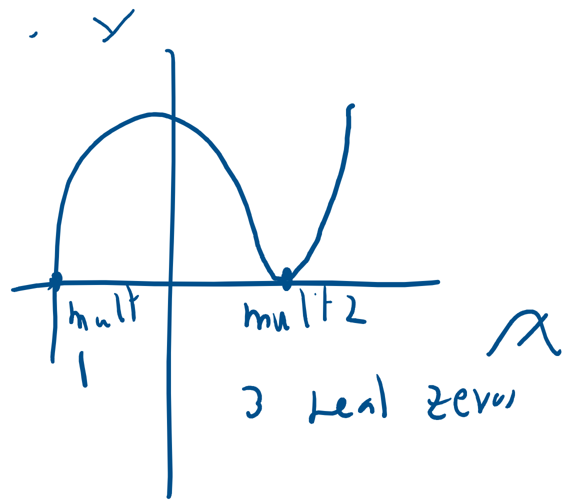
$\bar{z} = \bar{3} = 3 - 0i = 3 \in \mathbb{R}$

memorize

Theorem 3.16. Real Factorization Theorem: Suppose f is a polynomial function with real number coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors which give the nonreal zeros of f.



3 real zero



1 real zero
2 complex zero