09-30-25 MTH 167-002N

3 Polynomial Functions

3.1 Graphs of Polynomials

3.1.1 Exercises

page 246 (258): 3, 7, 13, 21, 27

3.2 The Factor Theorem and the Remainder Theorem

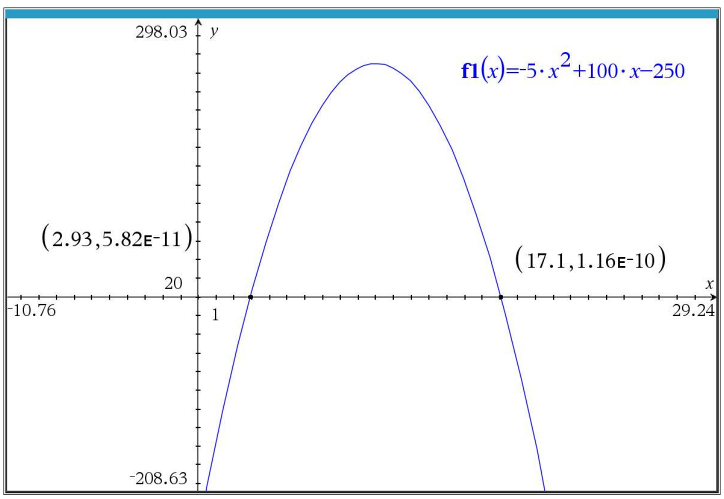
3.2.1 Exercises

page 265 (277): 1, 3, 9, 21, 35, 42

2.4: 36

36. The height h in feet of a model rocket above the ground t seconds after lift-off is given by $h(t) = -5t^2 + 100t$, for $0 \leq t \leq 20$. When is the rocket at least 250 feet off the ground? Round your answer to two decimal places.

Solve $h(t) \geq 250$



The rocket is at least 250 off the ground between 2.93 seconds and 17.1 seconds after lift-off.

After class notes

Solve this problem analytically to obtain the exact answer.

$$-5t^2 + 100t \geq 250$$

$$-5t^2 + 100t - 250 \geq 0$$

$$\frac{-5t^2}{-5} + \frac{100t}{-5} - \frac{250}{-5} \leq \frac{0}{-5}$$

$$t^2 - 20t + 50 \leq 0$$

First solve $t^2 - 20t + 50 = 0$

$t^2 - 20t + 50 = 0$

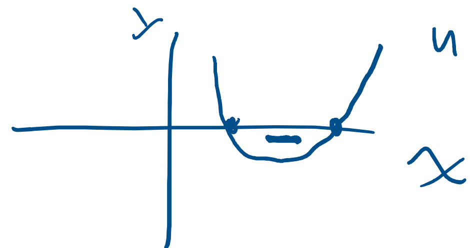
First solve $t^2 - 20t + 50 = 0$

$$t = \frac{20 \pm \sqrt{400 - 200}}{2}$$

$$t = \frac{20 \pm \sqrt{200}}{2} = \frac{20 \pm \sqrt{2 \cdot 100}}{2} = \frac{20 \pm \sqrt{100}\sqrt{2}}{2}$$

$$t = \frac{20 \pm 10\sqrt{2}}{2} = \boxed{10 \pm 5\sqrt{2}}$$

10-5*sqrt(2)=2.928932188134524
10+5*sqrt(2)=17.07106781186548



upward opening parabola

Rocket is at least 250 feet off the ground between $10 - 5\sqrt{2}$ seconds and $10 + 5\sqrt{2}$ seconds after lift-off.

3.2
Supplied

Theorem 3.4. Polynomial Division: Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

$$p(x) = d(x)\,q(x) + r(x)$$

polynomial divisor quotient remainder

Supplied

Theorem 3.5. The Remainder Theorem: Suppose p is a polynomial of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$.

Thm 3.4 $\Rightarrow p(x) = d(x)\,q(x) + r(x)$

$p(x) = (x - c)\,q(x) + r(x)$ Prove $p(c) = r(x)$

$p(c) = (c - c)\,q(c) + r(c)$

$p(c) = (0)\,q(c) + r(c)$ $[x - 1 = x^{(1)} - 1$

$p(c) = r(c)$

$p(c) = (0)\,q(c) + r(c)$

$p(c) = r(c)$

$\left[\, x-1 = x^{(1)}-1 \right.$

Thm 3.4 $\Rightarrow r(x) = 0$ or $\deg r(x) < \deg(x-1)$
$= 1$

$\Rightarrow \boxed{r(x) = 0}$ or $\deg r(x) = 0$

$\Rightarrow r(x) = w \neq 0$

If $r(x) = 0$ for all x then in particular $r(c) = 0$

$\Rightarrow \boxed{p(c) = r(c) = 0}$

If $r(x) = w \neq 0$, then $w = r(c)$

$\Rightarrow p(c) = r(c)$

Memorize

Theorem 3.6. The Factor Theorem: Suppose p is a nonzero polynomial. The real number c is a zero of p if and only if $(x-c)$ is a factor of $p(x)$.

Assume $x-c$ is a factor of $p(x)$

$\Rightarrow p(x) = (x-c)\,q(x) + r(x)$
with $r(x) = 0$

$\Rightarrow p(x) = (x-c)\,q(x)$

$\Rightarrow p(c) = (c-c)\,q(c)$
$= (0)\,q(c) = 0$

$\therefore c$ is a zero of $p(x)$

Memorize

Theorem 3.7. Suppose f is a polynomial of degree $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

$f(x) = (x-1)^2 = x^2 - 2x + 1$

deg of $f = 2 \geq 1$

deg of $f = 2 \geq 1$

f has at most 2 real zeros, counting multiplicities

zero $x = 1$

mult 2

$f(x) = x^2 + 1$ deg 2

at most 2 real zeros

$x^2 + 1 = 0$

$x^2 = -1$

$x = \pm i$, no real solution

Memorize

Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x - c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

$(x^3 + 2x + 5) \div (x - 1)$

$$
\begin{array}{r|l}
 & x^2 + x + 3 \quad + \frac{8}{x-1} \\
x - 1 & x^3 + 0 \cdot x^2 + 2x + 5 \\
 & x^3 - x^2 \\
 & \quad x^2 + 2x \\
 & \quad x^2 - x \\
 & \qquad 3x + 5 \\
 & \qquad 3x - 3 \\
 & \qquad - \\
\end{array}
$$

$$\begin{array}{r} 3x-3 \\ \hline 8 \end{array}$$

check $(x-1)\left(x^2+x+3+\frac{8}{x-1}\right)$

$$= x^3+x^2+3x$$
$$-x^2-x-3$$
$$+\underbrace{(x-1)\left(\frac{8}{x-1}\right)}_{=8}$$

$$= x^3+2x-3+8$$
$$= x^3+2x+5 \quad \checkmark$$

$$p(x) = d(x)\,q(x) + r(x)$$

$$x^3+2x+5 = (x-1)(x^2+x+3)+8$$

$$\frac{x^3+2x+5}{x-1} = \frac{(x-1)(x^2+x+3)}{x-1} + \frac{8}{x-1}$$

$$\frac{x^3+2x+5}{x-1} = x^2+x+3+\frac{8}{x-1}$$

Synthetic division only applies with linear divisors

$$(x^3+2x+5)\div(x-1)$$

mult.

$$\begin{array}{c|cccc} 1 & 1 & 0 & 2 & 5 \\ & & + & & \\ & \downarrow & 1 & 1 & 3 \\ \hline & 1 & 1 & 3 & (8) \end{array}$$

$$1 \cdot x^2 + 1 \cdot x + 3 + \frac{8}{x-1}$$

$$x^2 + x + 3 + \frac{8}{x-1}$$

Gemini

2.5 Flash

Can you explain why synthetic division of polynomials works?

Google Search

Synthetic division is a **shorthand method** for polynomial long division, designed specifically for the case when the divisor is a **linear binomial** of the form $(x - c)$. Its effectiveness stems from the fact that it automates and simplifies the repetitive steps in long division by focusing only on the coefficients.

Here's a breakdown of why it works:

1. Coefficient Only

Polynomial long division involves a lot of writing out variables and powers of x. Since you are dividing by a linear term, the power of x in each term of the quotient is always one less than the corresponding power in the dividend.

- Synthetic division eliminates the need to write the variables (x^n, x^{n-1}, etc.), focusing only on the **coefficients** of the terms in the polynomial. The position of each number automatically indicates the power of x it corresponds to.

2. Simplified Subtraction

In polynomial long division, a key step is multiplying a term in the quotient by the divisor and then **subtracting** the result from the dividend. This subtraction often leads to sign errors.

- Synthetic division uses the **root** (c) of the divisor $(x-c)$, instead of the divisor itself. If you set $x-c=0$, you get $x=c$.
- By using c (instead of $-c$) for the multiplication step, the process naturally incorporates the necessary sign change, allowing all subsequent operations to be **addition** instead of subtraction. This simplifies the arithmetic and reduces errors.

3. Connection to the Remainder Theorem

The process of synthetic division is also mathematically equivalent to **Horner's method** for evaluating a polynomial, which is closely tied to the **Remainder Theorem**.

- The **Division Algorithm** for polynomials states that when a polynomial $P(x)$ is divided by a divisor $D(x)$, we get a quotient $Q(x)$ and a remainder $R(x)$ such that:

$$P(x) = D(x) \cdot Q(x) + R(x)$$

- If the divisor is $D(x) = x - c$, this becomes:

$$P(x) = (x-c) \cdot Q(x) + R$$

(The remainder R is a constant because the divisor is degree 1).

- The **Remainder Theorem** states that $P(c) = R$. If you plug c into the equation above:

$$P(c) = (c-c) \cdot Q(c) + R$$

$$P(c) = (0) \cdot Q(c) + R$$

$$P(c) = R$$

- Synthetic division, by using c and the steps of multiply-and-add, essentially **calculates** $P(c)$. Therefore, the final number (the remainder) generated by the synthetic division process is the value of the polynomial at $x = c$, and the preceding numbers are the coefficients of the quotient. If the remainder is zero, the **Factor Theorem** tells us that $(x-c)$ is a factor of the polynomial.

The fundamental reason synthetic division works is that it is a **condensed and optimized version of polynomial long division** when the divisor is a monic linear factor.

You can learn more about this by watching Why

synthetic division works.
Why synthetic division works | Polynomial and rational functions | Algebra II

3.2

In Exercises 31 - 40, you are given a polynomial and one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

34. $2x^3 - 3x^2 - 11x + 6, \;\; c = \frac{1}{2}$

$$\begin{array}{c|cccc} \frac{1}{2} & 2 & -3 & -11 & 6 \\ & & 1 & -1 & -6 \\ \hline & 2 & -2 & -12 & (0) \end{array}$$

$2x^3 - 3x^2 - 11x + 6 = (x - \frac{1}{2})(2x^2 - 2x - 12)$

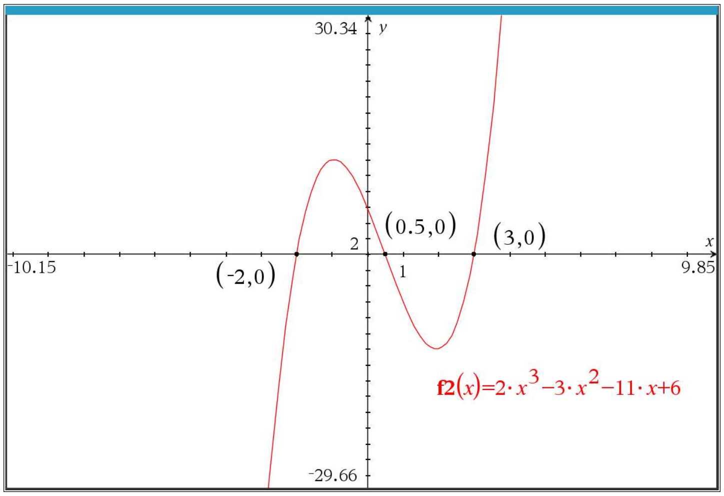
$x - \frac{1}{2}$ is a factor

solve $2x^2 - 2x - 12 = 0$

$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$\boxed{x = -2, 3}$



3.2

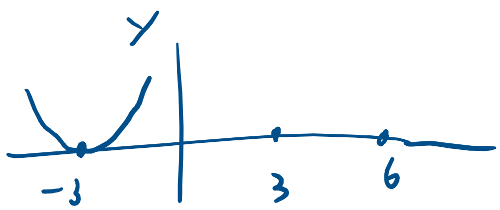
In Exercises 41 - 45, create a polynomial p which has the desired characteristics. You may leave the polynomial in factored form.

43.
- The solutions to $p(x) = 0$ are $x = \pm 3$ and $x = 6$
- The leading term of $p(x)$ is $7x^4$
- The point $(-3, 0)$ is a local minimum on the graph of $y = p(x)$.

$$p(x) = (x-3)^{1?}(x+3)^{2}(x-6)^{1?}\,(\,?\,)$$

multiplicitie add up to 4

no poly but constant



x+3 has mult even < 4
∴ mult 2

$$p(x) = a(x-3)(x+3)^2(x-6)$$

leading term $= ax^4 = 7x^4$

$$\boxed{a = 7}$$

$$p(x) = 7(x-3)(x+3)^2(x-6)$$

After class notes

$$(4x^3 + 5x^2 - x + 2) \div (x^2 + x + 1)$$

$$4x+1 + \frac{-6x+1}{\quad}$$

$$4x+1 + \frac{-6x+1}{x^2+x+1}$$

$$x^2+x+1 \overline{\big) 4x^3+5x^2-x+2}$$

$$\underline{4x^3+4x^2+4x}$$

$$x^2-5x+2$$

$$\underline{x^2+x+1}$$

$$-6x+1$$

Think

$$\frac{4x^3}{x^2} = ?$$

check: $(x^2+x+1)\left(4x+1+\frac{-6x+1}{x^2+x+1}\right)$

$$4x^3+x^2$$

$$+4x^2+x$$

$$+4x+1$$

$$(x^2+x+1)\left(\frac{-6x+1}{x^2+x+1}\right)$$

$$4x^3+5x^2+5x+1-6x+1$$

$$4x^3+5x^2-x+2 \quad \checkmark$$