04-03-25 MTH 161-004N

3.3 Real Zeros of Polynomials
3.3.3: Exercises
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3.4 Complex Zeros and the Fundamental Theorem of Algebra
3.4.1 Exercises
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Memorize

Definition 3.4. The imaginary unit i satisfies the two following properties

1. $i^2 = -1$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

$\sqrt{-c} = \sqrt{-1}\sqrt{c}$

$c \quad , c \geq 0$

$= i\sqrt{c}$

Memorize

Definition 3.5. A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit.

$\mathbb{C} = \mathbb{C} =$ set of all complex numbers

$\mathbb{R} = \mathbb{R} =$ set of all real numbers

$\mathbb{C} = \{a+bi \mid a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1\}$

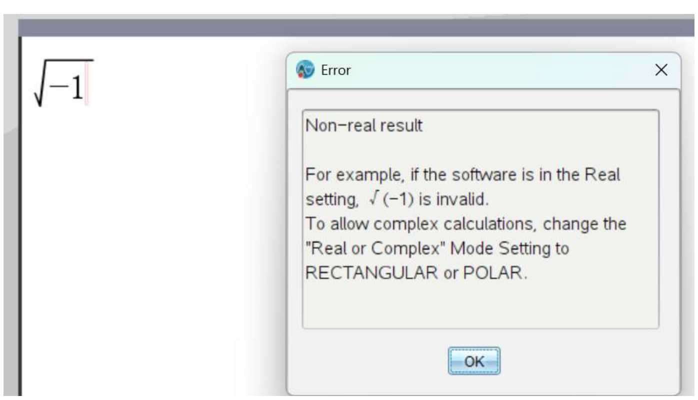
$(2+3i) + (5-i) = (2+5) + (3-1)i$

$$(2+3i)+(5-i) = (2+5) + (3-1)i$$
$$= 7+2i$$

Def $(a+bi)+(c+di)$
$$= (a+b) + (b+d)i$$

Def $(a+bi)-(c+di)$
$$= (a-c) + (b-d)i$$

$$(2+3i)(4-2i)$$
$$= 8 + 12i - 4i - 6i^2$$
$$= 8 + 8i - 6(-1)$$
$$= \boxed{14+8i}$$



$\mathbb{R} \subseteq \mathbb{C}$

$n \in \mathbb{R} \Rightarrow n = n + 0i \in \mathbb{C}$

memorize

Def Let $z = a + bi \in \mathbb{C}$

conjugate of $z = \bar{z} = a - bi$

$$\frac{2-i}{3+i} = \left(\frac{2-i}{3+i}\right)\left(\frac{3-i}{3-i}\right)$$

$$= \frac{(2-i)(3-i)}{(3+i)(3-i)}$$

$$= \frac{6 - 3i - 2i + i^2}{3^2 - i^2}$$

$$= \frac{6 - 5i - 1}{9 - (-1)}$$

$$= \frac{5 - 5i}{10}$$

$$= \frac{1-i}{2} = \frac{1}{2} - \left(\frac{1}{2}\right)i \in \mathbb{C}$$

Remember

$(a+b)(a-b)$

$= a^2 - b^2$

want $b^2 - 4ac < 0$

want $b^2 - 4ac < 0$

Let $b=3$, $a=4$, $c=1$

$9-(4)(4)(1)$

$=9-16=-7<0$

Let $f(x) = ax^2 + bx + c$

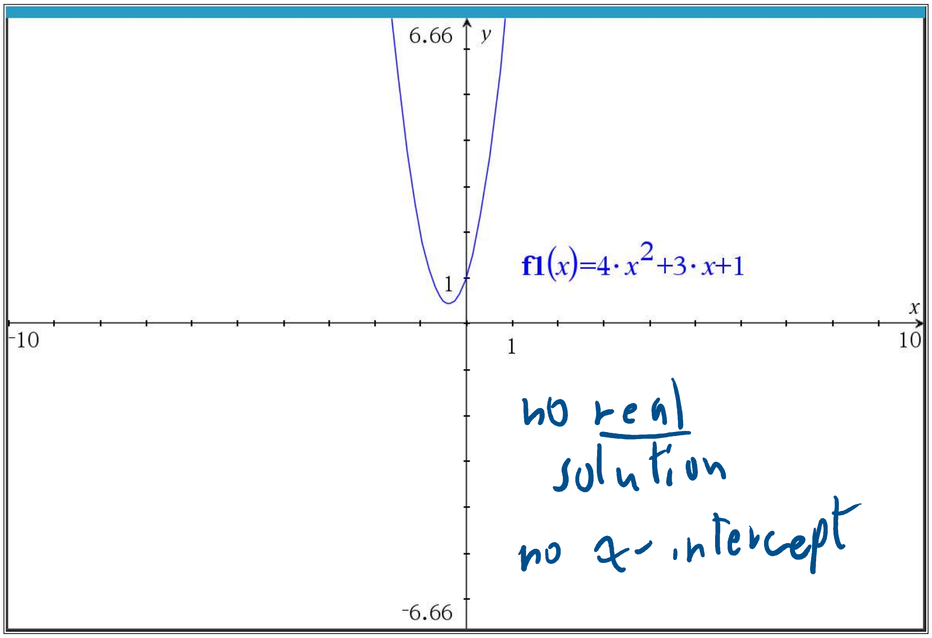
$f(x) = 4x^2 + 3x + 1$

Find zero of $f(x)$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{-7}}{8}$$

$$\boxed{x = -\frac{3}{8} \pm \left(\frac{1}{8}\right)\left(\sqrt{7}\right) i}$$



supplied

Theorem 3.13. The Fundamental Theorem of Algebra: Suppose f is a polynomial function with complex number coefficients of degree $n \geq 1$, then f has at least one complex zero.

Supplied

Theorem 3.14. Complex Factorization Theorem: Suppose f is a polynomial function with complex number coefficients. If the degree of f is n and $n \geq 1$, then f has exactly n complex zeros, counting multiplicity. If $z_1, z_2, \ldots, z_k$ are the distinct zeros of f, with multiplicities m_1, $m_2, \ldots, m_k$, respectively, then $f(x) = a(x - z_1)^{m_1}(x - z_2)^{m_2} \cdots (x - z_k)^{m_k}$.

Memorize

Theorem 3.15. Conjugate Pairs Theorem: If f is a polynomial function with real number coefficients and z is a zero of f, then so is $\overline{z}$.

Memorize

Theorem 3.16. Real Factorization Theorem: Suppose f is a polynomial function with real number coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors which give the nonreal zeros of f.

3.4.1 Exercises

In Exercises 1 - 10, use the given complex numbers z and w to find and simplify the following. Write your answers in the form $a + bi$.

- $z+w$
- zw
- z^2
- $\dfrac{1}{z}$
- $\dfrac{z}{w}$
- $\dfrac{w}{z}$
- $\overline{z}$
- $z\overline{z}$
- $(\overline{z})^2$

4. $z = 4i$, $w = 2 - 2i$

$$z \cdot \overline{z} = (4i)(-4i) = -16i^2 = -16(-1) = \boxed{16}$$

3.4

In Exercises 27 - 48, find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.

28. $f(x) = x^2 - 2x + 5$

$$x = \frac{2 \pm \sqrt{4-(4)(1)(5)}}{2}$$

$$x = \frac{2 \pm \sqrt{4-20}}{2}$$

$$x = \frac{2 \pm \sqrt{-16}}{2}$$

$$x = \frac{2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$x = \frac{2 \pm 4i}{2}$$

$$\boxed{x = 1 \pm 2i}$$

$$\boxed{\begin{aligned} f(x) &= (x-(1+2i))(x-(1-2i)) \\ f(x) &= ((x-1)-2i)((x-1)+2i) \end{aligned}}$$

$$f(x) = ((x-1)^2-(2i)^2)$$

$$f(x) = (x-1)^2+4$$

Textbook answer

28. $f(x) = x^2 - 2x + 5 = (x-(1+2i))(x-(1-2i))$
Zeros: $x = 1 \pm 2i$

3.4

In Exercises 49 - 53, create a polynomial f with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.

53.
- The leading term of $f(x)$ is $-2x^3$
- $c = 2i$ is a zero
- $f(0) = -16$

$f(x) = -2x^3 +$ lower degree terms

$x-2i$ is a factor of $f(x)$

$c = -2i$ is a zero

$\Rightarrow x+2i$ is a factor of $f(x)$

$$f(x) = -2x^3 + ? = (?)(x-2i)(x+2i)$$

$$(?)(-2x)(x-2i)(x+2i)$$

$(?)(-2x)(x-2i)(x+2i)$

Now $f(0) = (?)(-2(0))(0-2i)(0+2i) = 0$

but we want $f(0) = -16$

Try $f(x) = (-2x)(x-2i)(x+2i)$

$= (-2x)(x^2-4i^2) - 16$

$= (-2x)(x^2+4) - 16$

$\boxed{f(x) = -2x^3 - 8x - 16}$ Wrong

Textbook answer

53. $f(x) = -2(x-2i)(x+2i)(x+2)$

Finish at home

$$\begin{array}{r|l} & -2x \qquad + \frac{16}{x^2+4} \\ x^2+4 & -2x^3 + 0x^2 - 8x + 16 \\ & -2x^3 \qquad\quad -8x \\ \hline & \qquad\qquad\qquad 16 \end{array}$$

$-2x \qquad -8x$

16

$\therefore x^2+4$ is not a factor

Try again