03-27-25 MTH 161-004N

3.1 Graphs of Polynomials
    3.1.1 Exercises
    page 246: 3, 7, 13, 21, 27

3.2 The Factor Theorem and the Remainder Theorem
    3.2.1 Exercises
    page 265: 1, 3, 9, 21, 35, 42

3.3 Real Zeros of Polynomials
    3.3.3: Exercises
    page 280: 1, 31, 37, 48

3.2
Long division of polynomials

$$(2x^3 + 5x^2 - 4x + 1) \div (x - 2)$$

$$
\begin{array}{r|l}
 & 2x^2 + 9x + 14 + \frac{29}{x-2} \\
x - 2 & 2x^3 + 5x^2 - 4x + 1 \\
 & 2x^3 - 4x^2 \\
\hline
 & \quad 9x^2 - 4x \\
 & \quad 9x^2 - 18x \\
\hline
 & \qquad 14x + 1 \\
 & \qquad 14x - 28 \\
\hline
 & \qquad\quad 29
\end{array}
$$

check $(x-2)\left(2x^2 + 9x + 14 + \frac{29}{x-2}\right)$

$= 2x^3 + 9x^2 + 14x$

$18x \; - 28$ $\quad \Big| \; (\cancel{x-2})\left(\frac{29}{\cancel{x-2}}\right)$

$$-4x^2 - 18x - 28$$

$$+29$$

$$(x-2)\left(\frac{29}{\cancel{x-2}}\right)$$

$$= 2x^3 + 5x^2 - 4x + 1 \quad \checkmark$$

Synthetic division of polynomials: only applies when the divisor is a linear function

Divide by $x-2$, write 2

$$\begin{array}{c|cccc}
2 & 2 & 5 & -4 & 1 \\
 & & 4 & 18 & 28 \\
\hline
 & 2 & 9 & 14 & (29)
\end{array}$$

multiply

add

remainder

$$\boxed{2x^2 + 9x + 14 + \frac{29}{x-2}}$$

Supplied

Theorem 3.4. Polynomial Division: Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

$$\underbrace{\overbrace{p(x)}^{\text{given}}}_{\text{polynomial}} = \underbrace{\overbrace{d(x)}^{\text{given}}}_{\text{divisor}}\;\underbrace{\overbrace{q(x)}^{\text{this exists}}}_{\text{quotient}} + \underbrace{\overbrace{r(x)}^{\text{this exists}}}_{\text{remainder}}$$

polynomial division quotient remainder

Memorize the theorem, not the proof

Theorem 3.5. The Remainder Theorem: Suppose p is a polynomial of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$.

Here $d(x) = x - c$

Thm 3.4 $\Rightarrow p(x) = (x-c)q(x) + r(x)$

We don't the polynomials q(x) and r(x), but we do know that they exist and they are unique

$$p(c) = (c-c)q(c) + r(c)$$

$$p(c) = 0 \cdot q(c) + r(c)$$

$$\boxed{p(c) = r(c)}$$

Thm 3.4 $\Rightarrow r(x) = 0$ for all x

or $\deg r < \deg(x-c) = 1$

$\Rightarrow \deg r = 0$

$\Rightarrow r(x) = \text{constant} \neq 0$

Case 1: $r(x) = 0$ all x

In particular $r(c) = 0$

$\Rightarrow p(c) = 0 = r(x)$ ✓

case 2: $r(x) = k \neq 0$

$\Rightarrow r(c) = k$

$\Rightarrow p(c) = k$

$\therefore p(c) = r(x)$ ✓

Memorize

Theorem 3.6. The Factor Theorem: Suppose p is a nonzero polynomial. The real number c is a zero of p if and only if $(x - c)$ is a factor of $p(x)$.

Definition 1.9. The zeros of a function f are the solutions to the equation $f(x) = 0$. In other words, x is a zero of f if and only if $(x, 0)$ is an x-intercept of the graph of $y = f(x)$.

proof: Assume $x - c$ is a factor of $p(x)$

Prove $p(c) = 0$

$p(x) = (x - c)\, q(x)$, $q(x) =$ some poly

$p(c) = (c - c)\, q(c) = 0 \cdot q(c) = 0$ ✓

Assume $\boxed{p(c) = 0}$

Prove $x - c$ is a factor of $p(x)$

Thm 3.4 $\boxed{p(x) = (x - c)\, q(x) + r(x)}$

we are given unique poly $q(x), r(x)$

$r(x) = 0$ or $\deg r(x) < \deg(x - c) = 1$

$\Rightarrow \deg r(x) = 0$

$\Rightarrow r(x) = k \neq 0$ constant

By remainder theorem, $\boxed{r(x) = p(c)}$

By remainder theorem, $\boxed{r(x) = p(c)}$ $\Rightarrow r(x) = k \neq 0$ constant

$\Rightarrow p(x) = (x-c)q(x) + p(c)$

Assumed $p(c) = 0$

$\Rightarrow p(x) = (x-c)q(x)$

$\therefore x-c$ is a factor of $p(x)$

memorize

Theorem 3.7. Suppose f is a polynomial of degree $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

$f(x) = (x-2)^2(x-3)$

$= (x^2-4x+4)(x-3)$

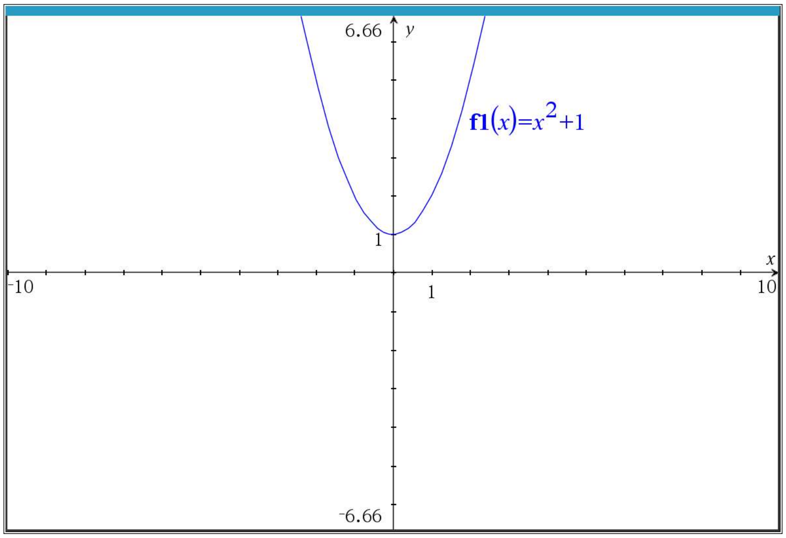
$= x^3 - 3x^2 - 4x^2 + 12x + 4x - 12$

$\boxed{f(x) = x^3 - 7x^2 + 16x - 12}$

Zero are 2 (mult 2), 3

$g(x) = x^2 + 1$

No real zero



True or False?

$x^{10} + 3x^7 - 4x^2 = 0$

has exactly 15 real zeros

False because $15 > 10$

Memorize

Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x - c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

3.3

Supplied

Theorem 3.8. Cauchy's Bound: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}$, $\frac{|a_1|}{|a_n|}$, $\ldots$, $\frac{|a_{n-1}|}{|a_n|}$. Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.

3

Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.

$$f(x) = 4x^3 - 10x^2 + x - 2$$

$a_0 = -2$

$a_1 = 1$

$a_2 = -10$

$a_3 = 4$

$$\left\{ \left|\frac{-2}{4}\right|, \left|\frac{1}{4}\right|, \left|\frac{-10}{4}\right| \right\}$$

$$= \left\{ \frac{1}{2}, \frac{1}{4}, \frac{5}{2} \right\}$$

$$M = \max\left\{ \frac{1}{2}, \frac{1}{4}, \frac{5}{2} \right\}$$

$$\boxed{M = \frac{5}{2}}$$

$$M + 1 = 1 + \frac{5}{2} = \frac{7}{2}$$

all real solution are in the interval $\left[-\frac{7}{2}, \frac{7}{2}\right]$